



Received: 31 December 2014
Accepted: 19 June 2015
Published: 12 August 2015

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Reviewing editor:
Lishan Liu, Qufu Normal University,
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Additional information is available at
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PURE MATHEMATICS | RESEARCH ARTICLE

On generalization of an integral inequality and its applications

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Abstract: In this paper, we give generalization of an integral inequality. We find its applications in fractional calculus by involving different kinds of fractional integral operators, for example Riemann–Liouville fractional integral, Caputo fractional derivative, Canavati fractional derivative and Widder derivative, Saigo fractional integral operator, etc.

Subjects: Advanced Mathematics; Mathematics & Statistics; Science

Keywords: monotone functions; integral inequalities; Widder derivatives; fractional derivatives and fractional integrals

AMS subject classifications: Primary 26A33; Secondary 26D15

1. Introduction

In the ocean of inequalities, integral inequalities received great attention by many scientists, for example mathematicians, physicists, and statisticians. Here, we want to pay our attention to an integral inequality by Mitrinović and Pečarić (1991).

THEOREM 1.1 Let $f_i: [0, \infty) \rightarrow \mathbb{R}$, $i = 1, 2, 3, 4$, be non-negative functions and let g be a real function which has the following representation

$$g(x) = \int_0^{\infty} k(x, t) dh(t), \quad (1.1)$$

where $k(x, t) \geq 0$, when $x \in [0, \infty)$, $t \in [0, \infty)$, and h is a non-decreasing function. If p, q are two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then

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I am working as Assistant Professor in the department of Mathematics COMSATS Institute of Information Technology, Attock Campus, Pakistan. I have completed my PhD in the subject of Mathematics from Abdus Salam School of Mathematical Sciences, Govt. College University, Lahore, Pakistan with specialization in Mathematical inequalities. My research interest area is Mathematical analysis, Functional analysis, Fractional Calculus, Mathematical Statistics. Recent paper is on generalization of an integral inequality and its applications in fractional calculus. Fractional integral inequalities are useful in establishing the uniqueness of solutions of certain partial differential equations also provides upper and lower bounds for the solutions of fractional boundary value problems.

PUBLIC INTEREST STATEMENT

Fractional calculus is as important as calculus and a lot of work has been published in the favor of fractional calculus. Fractional integral inequalities are useful in establishing the uniqueness of solutions of certain partial differential equations, and also provide upper and lower bounds for solutions of fractional boundary value problems. In this paper, an integral inequality is generalized and its applications in fractional calculus are found.

$$\int_0^\infty f_1(x)f_2(x)g(x)dx \leq C \left(\int_0^\infty f_3(x)g(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty f_4(x)g(x)dx \right)^{\frac{1}{q}} \tag{1.2}$$

where

$$C = \sup_t \left\{ \left(\int_0^\infty k(x,t)f_1(x)f_2(x)dx \right) \left(\int_0^\infty k(x,t)f_3(x)dx \right)^{\frac{-1}{p}} \left(\int_0^\infty k(x,t)f_4(x)dx \right)^{\frac{-1}{q}} \right\}.$$

Fractional integral inequalities are useful in establishing the uniqueness of solutions for certain fractional partial differential equations. They also provide upper and lower bounds for the solutions of fractional boundary value problems. These considerations have led various researchers in the field of integral inequalities to explore certain extensions and generalizations by involving fractional calculus operators (see Agarwal & Pang, 1995, Anastassiou, 2011, Andrić, Pečarić, & Perić, 2011, Widder, 1941).

In this paper, we are interested to give a generalization of integral inequality (Equation 1.2). Further, we give applications to fractional calculus; many researchers are working in this field and participate a lot in the theory of inequalities, for example, books in references Anastassiou (2009), Baleano, Diethelm, Scalas, and Trujillo (2012), Kilbas, Srivastava, and Trujillo, 2006, Bainov and Simeonov (1992), Widder (1941), also see for papers references Andrić, Barbir, Farid, and Pečarić (2014a, 2014b), Andrić, Pečarić, and Perić (2013a, 2013b), Andrić et al. (2011), Canavati (1987) and Farid and Pečarić (2012a, 2012b).

The organization of the paper is as follows: In Section 2, we give a generalization of integral inequality (Equation 1.2) and some remarks. In Section 3, we give applications for fractional integrals and fractional derivatives, and in Section 4, we give applications for Widder derivatives and linear differential operators. In the last Section 5, we give improvements of these results. In the whole paper, we suppose that all integrals exist.

2. Main results

Here, we give a generalization of integral inequality (Equation 1.2) and some remarks.

THEOREM 2.1 *Let $(\Omega_1, \Sigma_1, \mu_1)$, $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ -finite measures and $f_i: \Omega_2 \rightarrow \mathbb{R}$, $i = 1, 2, 3, 4$, be non-negative functions. Let g be the function having representation*

$$g(x) = \int_{\Omega_1} k(x,t)f(t)d\mu_1(t) \tag{2.1}$$

where $k: \Omega_2 \times \Omega_1 \rightarrow \mathbb{R}$ is a general non-negative kernel and $f: \Omega_1 \rightarrow \mathbb{R}$ be a real-valued function, and μ_2 is a non-decreasing function. If p, q are two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$, then

$$\int_{\Omega_2} f_1(x)f_2(x)g(x)d\mu_2(x) \leq C \left(\int_{\Omega_2} f_3(x)g(x)d\mu_2(x) \right)^{\frac{1}{p}} \left(\int_{\Omega_2} f_4(x)g(x)d\mu_2(x) \right)^{\frac{1}{q}} \tag{2.2}$$

where

$$C = \sup_{t \in \Omega_1} \left\{ \left(\int_{\Omega_2} k(x,t)f_1(x)f_2(x)d\mu_2(x) \right) \left(\int_{\Omega_2} k(x,t)f_3(x)d\mu_2(x) \right)^{\frac{-1}{p}} \left(\int_{\Omega_2} k(x,t)f_4(x)d\mu_2(x) \right)^{\frac{-1}{q}} \right\}.$$

Proof Using Equation 2.1, we have

$$\begin{aligned} \int_{\Omega_2} f_1(x)f_2(x)g(x)d\mu_2(x) &= \int_{\Omega_2} \int_{\Omega_1} f_1(x)f_2(x)k(x,t)f(t)d\mu_2(x)d\mu_1(t) \\ &= \int_{\Omega_1} \left(\int_{\Omega_2} k(x,t)f_1(x)f_2(x)d\mu_2(x) \right) f(t)d\mu_1(t) \leq C \left(\int_{\Omega_1} \int_{\Omega_2} k(x,t)f_3(x)d\mu_2(x) \right)^{\frac{1}{p}} \left(\int_{\Omega_2} k(x,t)f_4(x)d\mu_2(x) \right)^{\frac{1}{q}} f(t)d\mu_1(t) \\ &= C \int_{\Omega_1} \left(\frac{\lambda^p}{p} \int_{\Omega_2} k(x,t)f_3(x)d\mu_2(x) + \frac{\lambda^{-q}}{q} \int_{\Omega_2} k(x,t)f_4(x)d\mu_2(x) \right) f(t)d\mu_1(t), \end{aligned}$$

where

$$\lambda = \left(\int_{\Omega_2} k(x,t)f_3(x)d\mu_2(x) \right)^{-\frac{1}{pq}} \left(\int_{\Omega_2} k(x,t)f_4(x)d\mu_2(x) \right)^{\frac{1}{pq}}.$$

Therefore, we have

$$\begin{aligned} \int_{\Omega_2} f_1(x)f_2(x)g(x)d\mu_2(x) \\ \leq C \left(\frac{\lambda^p}{p} \int_{\Omega_2} f_3(x)g(x)d\mu_2(x) + \frac{\lambda^{-q}}{q} \int_{\Omega_2} f_4(x)g(x)d\mu_2(x) \right), \end{aligned}$$

while from value of λ we get (Equation 2.2). □

COROLLARY 2.2 If we set $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, then we get

$$\int_{\Omega_2} f_1(x)f_2(x)g(x)d\mu_2(x) \leq C \left(\int_{\Omega_2} f_1^p(x)g(x)d\mu_2(x) \right)^{\frac{1}{p}} \left(\int_{\Omega_2} f_2^q(x)g(x)d\mu_2(x) \right)^{\frac{1}{q}}$$

where

$$C = \sup_{t \in \Omega_1} \left\{ \left(\int_{\Omega_2} k(x,t)f_1(x)f_2(x)d\mu_2(x) \right) \left(\int_{\Omega_2} k(x,t)f_1^p(x)d\mu_2(x) \right)^{-\frac{1}{p}} \left(\int_{\Omega_2} k(x,t)f_2^q(x)d\mu_2(x) \right)^{\frac{1}{q}} \right\}.$$

COROLLARY 2.3 If we put $f(t) = 1$ and $\Omega_1 = \Omega_2 = [0, \infty)$ and replace $d\mu_1(t)$, $d\mu_2(x)$ by measures $dh(t)$, dx in Theorem 2, then we get Theorem 1.

Remark 2.4 If we put $f_3(x) = f_1^p(x)$, $f_4(x) = f_2^q(x)$, $f(t) = 1$, and $\Omega_1 = \Omega_2 = [0, \infty)$ and replace $d\mu_1(t)$, $d\mu_2(x)$ by measures $dh(t)$, dx , then we get a result of Volkov (1972).

3. Applications for fractional integrals

Let $[a, b]$, $-\infty < a < b < \infty$ be a finite interval on real axis \mathbb{R} . For $f \in L_1[a, b]$, the left-sided and right-sided **Riemann–Liouville fractional integrals** of order $\alpha > 0$ are defined by

$$J_{a+}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t) dt, \quad x > a,$$

$$J_{b-}^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t) dt, \quad x < b.$$

Here, Γ is the gamma function $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$.

THEOREM 3.1 Let p, q be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1, p > 1$. Then, for $\alpha > 0$, we have

$$\int_a^x f_1(x)f_2(x)J_{a+}^\alpha f(x)dx \leq C \left(\int_a^x f_3(x)J_{a+}^\alpha f(x)dx \right)^{\frac{1}{p}} \left(\int_a^x f_4(x)J_{a+}^\alpha f(x)dx \right)^{\frac{1}{q}} \tag{3.1}$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_a^x (x-t)^{\alpha-1} f_1(x)f_2(x)dx \right) \left(\int_a^x (x-t)^{\alpha-1} f_3(x)dx \right)^{\frac{1}{p}} \times \left(\int_a^x (x-t)^{\alpha-1} f_4(x)dx \right)^{\frac{1}{q}} \right\}.$$

Proof If we apply Theorem 2.1 with $\Omega_1 = \Omega_2 = (a, b), d\mu_1(x) = dx, d\mu_2(t) = dt$ and the kernel

$$k(x, t) = \begin{cases} \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} & a \leq t \leq x \\ 0 & x < t \leq b, \end{cases}$$

then g becomes $J_{a+}^\alpha f$ and we get the inequality (Equation 3.1). □

COROLLARY 3.2 If we set $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, then we get

$$\int_a^x f_1(x)f_2(x)J_{a+}^\alpha f(x)dx \leq C \left(\int_a^x f_1^p(x)J_{a+}^\alpha f(x)dx \right)^{\frac{1}{p}} \left(\int_a^x f_2^q(x)J_{a+}^\alpha f(x)dx \right)^{\frac{1}{q}}$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_a^x (x-t)^{\alpha-1} f_1(x)f_2(x)dx \right) \left(\int_a^x (x-t)^{\alpha-1} f_1^p(x)dx \right)^{\frac{1}{p}} \left(\int_a^x (x-t)^{\alpha-1} f_2^q(x)dx \right)^{\frac{1}{q}} \right\}.$$

THEOREM 3.3 Let p, q be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1, p > 1$. Then, for $\alpha > 0$, we have

$$\int_x^b f_1(x)f_2(x)J_{a-}^\alpha f(x)dx \leq C \left(\int_x^b f_3(x)J_{a-}^\alpha f(x)dx \right)^{\frac{1}{p}} \left(\int_x^b f_4(x)J_{a-}^\alpha f(x)dx \right)^{\frac{1}{q}} \tag{3.2}$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_x^b (t-x)^{\alpha-1} f_1(x)f_2(x)dx \right) \left(\int_x^b (t-x)^{\alpha-1} f_3(x)dx \right)^{\frac{1}{p}} \left(\int_x^b (t-x)^{\alpha-1} f_4(x)dx \right)^{\frac{1}{q}} \right\}.$$

Proof Applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (a, b), d\mu_1(x) = dx, d\mu_2(t) = dt$ and the kernel

$$k(x, t) = \begin{cases} \frac{(t-x)^{\alpha-1}}{\Gamma(\alpha)} & x \leq t \leq b \\ 0 & a < t \leq x, \end{cases}$$

then g becomes $J_{a-}^\alpha f$ and we get the inequality (Equation 3.2). □

COROLLARY 3.4 If we set $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, then we get

$$\int_x^b f_1(x)f_2(x)J_{a-}^\alpha f(x)dx \leq C \left(\int_x^b f_1^p(x)J_{a-}^\alpha f(x)dx \right)^{\frac{1}{p}} \left(\int_x^b f_2^q(x)J_{a-}^\alpha f(x)dx \right)^{\frac{1}{q}}$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_x^b (t-x)^{\alpha-1} f_1(x)f_2(x)dx \right) \left(\int_x^b (t-x)^{\alpha-1} f_1^p(x)dx \right)^{-\frac{1}{p}} \left(\int_x^b (t-x)^{\alpha-1} f_2^q(x)dx \right)^{\frac{1}{q}} \right\}.$$

Next, we observe the Caputo fractional derivatives (for details see Kilbas et al., 2006, Section 2.4, also Anastassiou, 2009, p. 449; Baleano et al., 2012, p. 16) for $\alpha \geq 0$ define n as

$$n = [\alpha] + 1, \text{ for } \alpha \notin \mathbb{N}_0; \quad n = [\alpha], \text{ for } \alpha \in \mathbb{N}_0, \tag{3.3}$$

where $[\cdot]$ is the integral part. For $f \in AC^n[a, b]$ the left-sided and right-sided Caputo fractional derivatives of order α are defined by

$${}^C D_{a+}^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{f^{(n)}(t)}{(x-t)^{\alpha-n+1}} dt, \tag{3.4}$$

and

$${}^C D_{a-}^\alpha f(x) = \frac{(-1)^n}{\Gamma(n-\alpha)} \int_x^b \frac{f^{(n)}(t)}{(t-x)^{\alpha-n-1}} dt. \tag{3.5}$$

THEOREM 3.5 Let p, q be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1, p > 1$. Then, for $f \in AC^n[a, b], \alpha \geq 0$, and n defined in Equation 3.3, we have

$$\int_a^x f_1(x)f_2(x) {}^C D_{a+}^\alpha f(x)dx \leq C \left(\int_a^x f_3(x) {}^C D_{a+}^\alpha f(x)dx \right)^{\frac{1}{p}} \left(\int_a^x f_4(x) {}^C D_{a+}^\alpha f(x)dx \right)^{\frac{1}{q}} \tag{3.6}$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_a^x (x-t)^{n-\alpha-1} f_1(x)f_2(x)dx \right) \left(\int_a^x (x-t)^{n-\alpha-1} f_3(x)dx \right)^{-\frac{1}{p}} \times \left(\int_a^x (x-t)^{n-\alpha-1} f_4(x)dx \right)^{\frac{1}{q}} \right\}.$$

Proof Applying Theorem 2 with $\Omega_1 = \Omega_2 = (a, b), d\mu_1(x) = dx, d\mu_2(t) = dt$ and the kernel

$$k(x, t) = \begin{cases} \frac{(x-t)^{n-\alpha-1}}{\Gamma(n-\alpha)} & a \leq t \leq x \\ 0 & x < t \leq b, \end{cases}$$

and replacing f by $f^{(n)}, g$ becomes ${}^C D_{a+}^\alpha f$ and we get the inequality (Equation 3.7). □

COROLLARY 3.6 If we set $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, then we get

$$\int_a^x f_1(x)f_2(x) {}^C D_{a+}^\alpha f(x)dx \leq C \left(\int_a^x f_1^p(x) {}^C D_{a+}^\alpha f(x)dx \right)^{\frac{1}{p}} \left(\int_a^x f_2^q(x) {}^C D_{a+}^\alpha f(x)dx \right)^{\frac{1}{q}}$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_a^x (x-t)^{n-\alpha-1} f_1(x) f_2(x) dx \right) \left(\int_a^x (x-t)^{n-\alpha-1} f_1^p(x) dx \right)^{\frac{-1}{p}} \right. \\ \left. \times \left(\int_a^x (x-t)^{n-\alpha-1} f_2^q(x) dx \right)^{\frac{-1}{q}} \right\}.$$

THEOREM 3.7 Let p, q be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1, p > 1$. Then, for $f \in AC^n[a, b], \alpha \geq 0$, and n defined in Equation 3.3, we have

$$\int_x^b f_1(x) f_2(x) {}^C D_{a-}^\alpha f(x) dx \leq C \left(\int_x^b f_3(x) {}^C D_{a-}^\alpha f(x) dx \right)^{\frac{1}{p}} \left(\int_x^b f_4(x) {}^C D_{a-}^\alpha f(x) dx \right)^{\frac{1}{q}} \tag{3.7}$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_x^b (x-t)^{n-\alpha-1} f_1(x) f_2(x) dx \right) \left(\int_x^b (x-t)^{n-\alpha-1} f_3(x) dx \right)^{\frac{-1}{p}} \right. \\ \left. \times \left(\int_x^b (x-t)^{n-\alpha-1} f_4(x) dx \right)^{\frac{-1}{q}} \right\}.$$

Proof Applying Theorem 2 with $\Omega_1 = \Omega_2 = (a, b), d\mu_1(x) = dx, d\mu_2(t) = dt$ and the kernel

$$k(x, t) = \begin{cases} \frac{(-1)^n (t-x)^{n-\alpha-1}}{\Gamma(n-\alpha)} & x \leq t \leq b \\ 0 & 0 < t \leq x, \end{cases}$$

and replacing f by $f^{(n)}, g$ becomes ${}^C D_{a-}^\alpha f$ and we get the inequality (Equation 3.7). □

COROLLARY 3.8 If we set $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, then we get

$$\int_x^b f_1(x) f_2(x) {}^C D_{a-}^\alpha f(x) dx \leq C \left(\int_x^b f_1^p(x) {}^C D_{a-}^\alpha f(x) dx \right)^{\frac{1}{p}} \left(\int_x^b f_2^q(x) {}^C D_{a+}^\alpha f(x) dx \right)^{\frac{1}{q}}$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_x^b (t-x)^{n-\alpha-1} f_1(x) f_2(x) dx \right) \left(\int_x^b (t-x)^{n-\alpha-1} f_1^p(x) dx \right)^{\frac{-1}{p}} \right. \\ \left. \times \left(\int_x^b (t-x)^{n-\alpha-1} f_2^q(x) dx \right)^{\frac{-1}{q}} \right\}.$$

We continue with extensions that require composition identities for the left-sided Caputo fractional derivatives, given in Andrić et al. (2013b).

LEMMA 3.9 Let $\beta > \alpha \geq 0, m$ and n are given by Equation 3.3 for β and α , respectively. Let $f \in AC^m[a, b]$ be such that $f^{(i)}(a) = 0$ for $i = n, n + 1, \dots, m - 1$. Let ${}^C D_{a+}^\beta f, {}^C D_{a+}^\alpha f \in L_1[a, b]$. Then,

$${}^C D_{a+}^\alpha f(x) = \frac{1}{\Gamma(\beta - \alpha)} \int_a^x (x-t)^{\beta-\alpha-1} {}^C D_{a+}^\beta f(t) dt, \quad x \in [a, b].$$

THEOREM 3.10 Let p, q be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1, p > 1$. Let $\beta > \alpha \geq 0, m$ and n are given by Equation 3.3 for β and α , respectively. Let $f \in AC^m[a, b]$ be such that $f^{(i)}(a) = 0$ for $i = n, n + 1, \dots, m - 1$. Let ${}^C D_{a+}^\beta f, {}^C D_{a+}^\alpha f \in L_1[a, b]$. Then, we have

$$\int_a^x f_1(x) f_2(x) {}^C D_{a+}^\alpha f(x) dx \leq C \left(\int_a^x f_3(x) {}^C D_{a+}^\alpha f(x) dx \right)^{\frac{1}{p}} \left(\int_a^x f_4(x) {}^C D_{a+}^\alpha f(x) dx \right)^{\frac{1}{q}} \tag{3.8}$$

where

$$C = \sup_{t \in [a, b]} \left\{ \left(\int_a^x (x-t)^{\beta-\alpha-1} f_1(x) f_2(x) dx \right) \left(\int_a^x (x-t)^{\beta-\alpha-1} f_3(x) dx \right)^{\frac{1}{p}} \left(\int_a^x (x-t)^{\beta-\alpha-1} f_4(x) dx \right)^{\frac{1}{q}} \right\}.$$

Proof Applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (a, b), d\mu_1(x) = dx, d\mu_2(t) = dt$ and the kernel

$$k(x, t) = \begin{cases} \frac{(x-t)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} & a \leq t \leq x \\ 0 & x < t \leq b, \end{cases}$$

and replacing f by ${}^C D_{a+}^\beta f, g$ becomes ${}^C D_{a+}^\alpha f$ and we get the inequality (Equation 3.8). □

COROLLARY 3.11 If we set $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, then we get

$$\int_a^x f_1(x) f_2(x) {}^C D_{a+}^\alpha f(x) dx \leq C \left(\int_a^x f_1^p(x) {}^C D_{a+}^\alpha f(x) dx \right)^{\frac{1}{p}} \left(\int_a^x f_2^q(x) {}^C D_{a+}^\alpha f(x) dx \right)^{\frac{1}{q}}$$

where

$$C = \sup_{t \in [a, b]} \left\{ \left(\int_a^x (x-t)^{\beta-\alpha-1} f_1(x) f_2(x) dx \right) \left(\int_a^x (x-t)^{\beta-\alpha-1} f_1^p(x) dx \right)^{\frac{1}{p}} \left(\int_a^x (x-t)^{\beta-\alpha-1} f_2^q(x) dx \right)^{\frac{1}{q}} \right\}.$$

Remark 3.12 Using Theorem 2.1 and composition identities for the right-sided Caputo fractional derivatives given in Andrić et al. (2013b, Theorem 2.2), similar results can be stated and proved for the right-sided Caputo fractional derivatives (for details see Andrić et al., 2014a; Farid, & Pečarić, 2012b).

Results given for the Caputo fractional derivatives can be analogously done for two other types of fractional derivative that we observe: Canavati type and Riemann–Liouville type. Here, as an example inequality for each type of fractional derivatives, we give inequality analogous to the Equation 3.8 obtained with composition identity for the left-sided fractional derivatives. Proofs are omitted.

For more details on the **Canavati fractional derivatives**, see Canavati (1987): we consider subspace $C_{a+}^\alpha[a, b]$ defined by

$$C_{a+}^\alpha[a, b] = \{f \in C^{n-1}[a, b]: J_{a+}^{n-\alpha} f^{(n-1)} \in C^1[a, b]\}.$$

For $f \in C_{a+}^\alpha[a, b]$, the left-sided Canavati fractional derivative of order α is defined by

$${}^C D_{a+}^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{n-\alpha-1} f^{(n-1)}(t) dt = \frac{d}{dx} J_{a+}^{n-\alpha} f^{(n-1)}(x).$$

Composition identity for the left-sided Canavati fractional derivatives is given in Andrić et al. (2011):

LEMMA 3.13 Let $\beta > \alpha > 0$, $m = [\beta] + 1$, and $n = [\alpha] + 1$. Let $f \in C_{\alpha+}^{\beta}[a, b]$ be such that $f^{(i)}(a) = 0$ for $i = n - 1, n, \dots, m - 2$. Then, $f \in C_{\alpha+}^{\alpha}[a, b]$ and

$${}^{\tilde{C}}D_{\alpha+}^{\alpha} f(x) = \frac{1}{\Gamma(\beta - \alpha)} \int_a^x (x - t)^{\beta - \alpha - 1} {}^{\tilde{C}}D_{\alpha+}^{\beta} f(t) dt, \quad x \in [a, b].$$

THEOREM 3.14 Let p, q be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. Let $\beta > \alpha \geq 0$, m and n given by Equation 3.3 for β and α , respectively. Let $f \in C_{\alpha+}^{\beta}[a, b]$ be such that $f^{(i)}(a) = 0$ for $i = n - 1, n, \dots, m - 2$. Then, we have

$$\int_a^x f_1(x) f_2(x) {}^{\tilde{C}}D_{\alpha+}^{\alpha} f(x) dx \leq C \left(\int_a^x f_3(x) {}^{\tilde{C}}D_{\alpha+}^{\alpha} f(x) dx \right)^{\frac{1}{p}} \left(\int_a^x f_4(x) {}^{\tilde{C}}D_{\alpha+}^{\alpha} f(x) dx \right)^{\frac{1}{q}}$$

where

$$C = \sup_{t \in [a, b]} \left\{ \left(\int_a^x (x - t)^{\beta - \alpha - 1} f_1(x) f_2(x) dx \right) \left(\int_a^x (x - t)^{\beta - \alpha - 1} f_3(x) dx \right)^{\frac{1}{p}} \times \left(\int_a^x (x - t)^{\beta - \alpha - 1} f_4(x) dx \right)^{\frac{1}{q}} \right\}.$$

For more details on **Riemann–Liouville fractional derivatives**, see Kilbas et al. (2006), Section 2.1: for $f: [a, b] \rightarrow \mathbb{R}$, the left-sided Riemann–Liouville fractional derivative of order α is defined by

$$D_{\alpha+}^{\alpha} f(x) = \frac{1}{\Gamma(n - \alpha)} \frac{d^n}{dx^n} \int_a^x (x - t)^{n - \alpha - 1} f(t) dt = \frac{d^n}{dx^n} J_{\alpha+}^{n - \alpha} f(x).$$

The following lemma summarizes conditions in the composition identity for the left-sided Riemann–Liouville fractional derivatives (for details see Andrić et al., 2013a).

LEMMA 3.15 Let $\beta > \alpha \geq 0$, $m = [\beta] + 1$, and $n = [\alpha] + 1$. The composition identity

$$D_{\alpha+}^{\alpha} f(x) = \frac{1}{\Gamma(\beta - \alpha)} \int_a^x (x - t)^{\beta - \alpha - 1} D_{\alpha+}^{\beta} f(t) dt, \quad x \in [a, b],$$

is valid if one of the following conditions holds:

- (i) $f \in J_{\alpha+}^{\beta}(L_1[a, b]) = \{f: f = J_{\alpha+}^{\beta} \varphi, \varphi \in L_1[a, b]\}$.
- (ii) $J_{\alpha+}^{m - \beta} f \in AC^m[a, b]$ and $D_{\alpha+}^{\beta - k} f(a) = 0$ for $k = 1, \dots, m$.
- (iii) $D_{\alpha+}^{\beta - 1} f \in AC[a, b]$, $D_{\alpha+}^{\beta - k} f \in C[a, b]$, and $D_{\alpha+}^{\beta - k} f(a) = 0$ for $k = 1, \dots, m$.
- (iv) $f \in AC^m[a, b]$, $D_{\alpha+}^{\beta} f, D_{\alpha+}^{\alpha} f \in L_1[a, b]$, $\beta - \alpha \notin \mathbb{N}$, $D_{\alpha+}^{\beta - k} f(a) = 0$ for $k = 1, \dots, m$ and $D_{\alpha+}^{\alpha - k} f(a) = 0$ for $k = 1, \dots, n$.
- (v) $f \in AC^m[a, b]$, $D_{\alpha+}^{\beta} f, D_{\alpha+}^{\alpha} f \in L_1[a, b]$, $\beta - \alpha = l \in \mathbb{N}$, $D_{\alpha+}^{\beta - k} f(a) = 0$ for $k = 1, \dots, l$.
- (vi) $f \in AC^m[a, b]$, $D_{\alpha+}^{\beta} f, D_{\alpha+}^{\alpha} f \in L_1[a, b]$, and $f^{(k)}(a) = 0$ for $k = 0, \dots, m - 2$.
- (vii) $f \in AC^m[a, b]$, $D_{\alpha+}^{\beta} f, D_{\alpha+}^{\alpha} f \in L_1[a, b]$, $\beta \notin \mathbb{N}$ and $D_{\alpha+}^{\beta - 1} f$ is bounded in a neighborhood of $m = a$.

THEOREM 3.16 Let p, q be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. Also, let $\beta > \alpha \geq 0$, $m = [\beta] + 1$, and $n = [\alpha] + 1$. Suppose that one of the conditions in (i)–(vii) in Lemma 3.15 holds for $\{\beta, \alpha, f\}$ and let $D_{\alpha+}^{\beta} f \in L_1[a, b]$, then we have

$$\int_a^x f_1(x) f_2(x) D_{\alpha+}^{\alpha} f(x) dx \leq C \left(\int_a^x f_3(x) D_{\alpha+}^{\alpha} f(x) dx \right)^{\frac{1}{p}} \left(\int_a^x f_4(x) D_{\alpha+}^{\alpha} f(x) dx \right)^{\frac{1}{q}}$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_a^x (x-t)^{\beta-\alpha-1} f_1(x) f_2(x) dx \right) \left(\int_a^x (x-t)^{\beta-\alpha-1} f_3(x) dx \right)^{\frac{-1}{p}} \right. \\ \left. \times \left(\int_a^x (x-t)^{\beta-\alpha-1} f_4(x) dx \right)^{\frac{-1}{q}} \right\}.$$

Next, we give the results for a generalized fractional integral operator, the **Saigo fractional integral operator** (for details see, Saigo, 1978).

Let $\alpha > 0, \beta, \eta \in \mathbb{R}$. Then, the Saigo fractional integrals $I_{a,t}^{\alpha,\beta,\eta}$ of order α for a real-valued continuous function f are defined by:

$$I_{a,x}^{\alpha,\beta,\eta} f(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha + \beta, -\eta; \alpha; 1 - \frac{t}{x}\right) f(t) dt, \quad x \in [a, b]. \tag{3.9}$$

where, the function ${}_2F_1(\dots)$ appearing as the kernel for operator (Equation 3.9) is the Gaussian hypergeometric function defined by

$${}_2F_1(a, b; c; t) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} t^n,$$

and $(a)_n$ is the Pochhammer symbol: $(a)_n = a(a+1) \dots (a+n-1), (a)_0 = 1$.

THEOREM 3.17 Let p, q be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1, p > 1$. Then, for $\alpha > 0$, we have

$$\int_a^x f_1(x) f_2(x) I_{a,x}^{\alpha,\beta,\eta} f(x) dx \leq C \left(\int_a^x f_3(x) I_{a,x}^{\alpha,\beta,\eta} f(x) dx \right)^{\frac{1}{p}} \left(\int_a^x f_4(x) I_{a,x}^{\alpha,\beta,\eta} f(x) dx \right)^{\frac{1}{q}} \tag{3.10}$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_a^x x^{-\alpha-\beta} {}_2F_1\left(\alpha + \beta, -\eta, \alpha; 1 - \frac{t}{x}\right) (x-t)^{\alpha-1} f_1(x) f_2(x) dx \right) \right. \\ \times \left(\int_a^x x^{-\alpha-\beta} {}_2F_1\left(\alpha + \beta, -\eta, \alpha; 1 - \frac{t}{x}\right) (x-t)^{\alpha-1} f_3(x) dx \right)^{\frac{-1}{p}} \\ \left. \times \left(\int_a^x x^{-\alpha-\beta} {}_2F_1\left(\alpha + \beta, -\eta, \alpha; 1 - \frac{t}{x}\right) (x-t)^{\alpha-1} f_4(x) dx \right)^{\frac{-1}{q}} \right\}.$$

Proof Applying Theorem 2 with $\Omega_1 = \Omega_2 = (a, b), d\mu_1(x) = dx, d\mu_2(t) = dt$ and the kernel g becomes $I_{a,x}^{\alpha,\beta,\eta} f$ and we get the inequality (Equation 3.10). \square

$$k(x, t) = \begin{cases} \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} {}_2F_1\left(\alpha + \beta, -\eta, \alpha; 1 - \frac{t}{x}\right) (x-t)^{\alpha-1}, & a \leq t \leq x; \\ 0, & x < t \leq b, \end{cases}$$

COROLLARY 3.18 If we set $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, then we get

$$\int_a^x f_1(x) f_2(x) I_{a,x}^{\alpha,\beta,\eta} f(x) dx \leq C \left(\int_a^x f_1^p(x) I_{a,x}^{\alpha,\beta,\eta} f(x) dx \right)^{\frac{1}{p}} \\ \times \left(\int_a^x f_2^q(x) I_{a,x}^{\alpha,\beta,\eta} f(x) dx \right)^{\frac{1}{q}}$$

where

$$C = \sup_{t \in [a,b]} \left\{ \left(\int_a^x x^{-\alpha-\beta} {}_2F_1(\alpha + \beta, -\eta, \alpha; 1 - \frac{t}{x})(x-t)^{\alpha-1} f_1(x) f_2(x) dx \right) \times \left(\int_a^x x^{-\alpha-\beta} {}_2F_1(\alpha + \beta, -\eta, \alpha; 1 - \frac{t}{x})(x-t)^{\alpha-1} f(x)^p dx \right)^{\frac{-1}{p}} \times \left(\int_a^x x^{-\alpha-\beta} {}_2F_1(\alpha + \beta, -\eta, \alpha; 1 - \frac{t}{x})(x-t)^{\alpha-1} f_2^q(x) dx \right)^{\frac{-1}{q}} \right\}.$$

Remark 3.19 If we put $\beta = -\alpha$ in Theorem 3.17, then we get the results for the right sided Riemann-Liouville fractional integral.

4. Applications for Widder derivatives and linear differential operators

In this section, we apply our results for Widder derivatives and linear differential operators. The following are taken from Widder (1928).

Let $f, u_0, u_1, \dots, u_n \in C^{n+1}([a, b])$, $n \geq 0$, and the Wronskians

$$W_i(x) := W[u_0(x), u_1(x), \dots, u_i(x)] := \begin{vmatrix} u_0(x) & u_1(x) & \dots & u_i(x) \\ u_0'(x) & u_1'(x) & \dots & u_i'(x) \\ \vdots & \vdots & \ddots & \vdots \\ u_0^{(i-1)}(x) & u_1^{(i-1)}(x) & \dots & u_i^{(i-1)}(x) \end{vmatrix}, \tag{4.1}$$

$i = 1, \dots, n$. Here, $W_0(x) = u_0(x)$. Assume $W_0(x) > 0$ over $[a, b]$, $i = 1, \dots, n$.

For $i \geq 0$, the differential operator of order i (Widder derivative):

$$L_i f(x) := \frac{W[u_0(x), u_1(x), \dots, u_{i-1}(x), f(x)]}{W_{i-1}(x)} \tag{4.2}$$

$i = 1, \dots, n + 1$; $L_0 f(x) := f(x), \forall x \in [a, b]$. Consider also

$$g_i(x, t) := \frac{1}{W_i(t)} \begin{vmatrix} u_0(t) & u_1(t) & \dots & u_i(t) \\ u_0'(t) & u_1'(t) & \dots & u_i'(t) \\ \vdots & \vdots & \ddots & \vdots \\ u_0^{(i-1)}(t) & u_1^{(i-1)}(t) & \dots & u_i^{(i-1)}(t) \\ u_0(x) & u_1(x) & \dots & u_i(x) \end{vmatrix}, \tag{4.3}$$

$i = 1, \dots, n$; $g_0(x, t) := \frac{u_0(x)}{u_0(t)}, \forall x, t \in [a, b]$.

Example 4.1 Sets of the form $\{u_0, u_1, \dots, u_n\}$ are $\{u_0, u_1, \dots, u_n\}, \{1, \sin x, -\cos x, -\sin 2x, \cos 2x, \dots, (-1)^n \sin nx, (-1)^n \cos nx\}$, etc.

We also mention the generalized Widder-Taylor’s formula (see Anastassiou, 2011; Widder, 1928).

THEOREM 4.2 Let the functions $f, u_0, u_1, \dots, u_n \in C^{n+1}([a, b])$, and the Wronskians $W_0(x), W_1(x), \dots, W_n(x) > 0$ on $[a, b]$, $x \in [a, b]$. Then, for $t \in [a, b]$ we have

$$f(x) = y(t) \frac{u_0(x)}{u_0(t)} + L_1 f(t) g_1(x, t) + \dots + L_n f(t) g_n(x, t) + R_n(x), \tag{4.4}$$

where

$$R_n(x) = \int_t^x g_n(x, s) L_{n+1} f(s) ds$$

For example (Widder, 1928), one could take $u_0(x) = c > 0$. If $u_i(x) = x^i, i = 0, 1, \dots, n$, defined on $[a, b]$, then

$$L_i y(t) = f^i(t) \text{ and } g_i(x, t) = \frac{(x-t)^i}{i!}, t \in [a, b].$$

We need the following result.

COROLLARY 4.3 By additionally assuming for fixed $x_0 \in [a, b]$ that $L_i f(x_0) = 0, i = 0, 1, \dots, n$, we get that

$$f(x) = \int_{x_0}^x g_n(x, s) L_{n+1} f(s) ds. \tag{4.5}$$

Note that all the results of this section are under the assumptions of Theorem 4.2 and Corollary 4.3.

THEOREM 4.4 Let p, q be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, for $p > 1$. Then, we have

$$\int_{x_0}^x f_1(x) f_2(x) f(x) dx \leq C \left(\int_{x_0}^x f_3(x) f(x) dx \right)^{\frac{1}{p}} \left(\int_{x_0}^x f_4(x) f(x) dx \right)^{\frac{1}{q}} \tag{4.6}$$

where

$$C = \sup_{t \in [a, b]} \left\{ \left(\int_{x_0}^x g_n(x, t) f_1(x) f_2(x) dx \right) \left(\int_{x_0}^x g_n(x, t) f_3(x) dx \right)^{\frac{-1}{p}} \times \left(\int_{x_0}^x g_n(x, t) f_4(x) dx \right)^{\frac{-1}{q}} \right\}.$$

Proof Applying Theorem 2 with $\Omega_1 = \Omega_2 = (a, b), d\mu_1(x) = dx, d\mu_2(t) = dt$ and the kernel $k(x, t)$ with particular kernel $g_n(x, t)$, also replacing f by $L_{n+1} f, g$ becomes f and we get the inequality (Equation 4.6). \square

COROLLARY 4.5 If we set $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, then we get

$$\int_{x_0}^x f_1(x) f_2(x) f(x) dx \leq C \left(\int_{x_0}^x f_1^p(x) f(x) dx \right)^{\frac{1}{p}} \left(\int_{x_0}^x f_2^q(x) f(x) dx \right)^{\frac{1}{q}}$$

where

$$C = \sup_{t \in [a, b]} \left\{ \left(\int_{x_0}^x g_n(x, t) f_1(x) f_2(x) dx \right) \left(\int_{x_0}^x g_n(x, t) f_1^p(x) dx \right)^{\frac{-1}{p}} \times \left(\int_{x_0}^x g_n(x, t) f_2^q(x) dx \right)^{\frac{-1}{q}} \right\}.$$

Onward, we follow Kreider, Kuller, and Perkins (1966, pp. 145–154).

Let I be a closed interval of \mathbb{R} . Let $a_i(x), i = 0, 1, \dots, n - 1 (n \in \mathbb{N}), h(x)$ be continuous functions on I and let $L = D^n + a_{n-1}(x)D^{n-1} + \dots + a_0(x)$ be a fixed linear differential operator on $C^n(I)$. Let $y_1(x), \dots, y_n(x)$ be a set of linear independent solutions to $Ly = 0$. Here, the associated Green's function for L is

$$H(x, t) = \frac{\begin{vmatrix} y_1(t) & \dots & y_n(t) \\ y_1'(t) & \dots & y_n'(t) \\ \vdots & & \vdots \\ y_1^{n-2}(t) & \dots & y_n^{n-2}(t) \\ y_1(x) & \dots & y_n(x) \end{vmatrix}}{\begin{vmatrix} y_1(t) & \dots & y_n(t) \\ y_1'(t) & \dots & y_n'(t) \\ \vdots & & \vdots \\ y_1^{n-2}(t) & \dots & y_n^{n-2}(t) \\ y_1(t) & \dots & y_n(t) \end{vmatrix}}, \tag{4.7}$$

which is a continuous function on I^2 . Consider fixed $x_0 \in I$, then

$$y(x) = \int_{x_0}^x H(x, t)h(t)dt, x \in I \tag{4.8}$$

is the unique solution to the initial value problem

THEOREM 4.6 Let p, q be two real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, for $p > 1$. Then, we have
 $Ly = h; y^{(i)}(x_0) = 0, i = 0, 1, \dots, n - 1.$ (4.9)

$$\int_{x_0}^x f_1(x)f_2(x)y(x)dx \leq C \left(\int_{x_0}^x f_3(x)y(x)dx \right)^{\frac{1}{p}} \left(\int_{x_0}^x f_4(x)y(x)dx \right)^{\frac{1}{q}} \tag{4.10}$$

where

$$C = \sup_{t \in (a,b)} \left\{ \left(\int_{x_0}^x H(x, t)f_1(x)f_2(x)dx \right) \left(\int_{x_0}^x H(x, t)f_3(x)dx \right)^{\frac{-1}{p}} \times \left(\int_{x_0}^x H(x, t)f_4(x)dx \right)^{\frac{-1}{q}} \right\}.$$

Proof Applying Theorem 2 with $\Omega_1 = \Omega_2 = (a, b), d\mu_1(x) = dx, d\mu_2(t) = dt$ and the kernel $k(x, t)$ with particular kernel $H(x, t)$, also replacing f by h, g becomes y and we get the inequality (Equation 4.10). □

COROLLARY 4.7 If we set $f_3(x) = f_1^p(x)$ and $f_4(x) = f_2^q(x)$, then we get

$$\int_{x_0}^x f_1(x)f_2(x)y(x)dx \leq C \left(\int_{x_0}^x f_1^p(x)y(x)dx \right)^{\frac{1}{p}} \left(\int_{x_0}^x f_2^q(x)y(x)dx \right)^{\frac{1}{q}}$$

where

$$C = \sup_{t \in (a,b)} \left\{ \left(\int_{x_0}^x H(x, t)f_1(x)f_2(x)dx \right) \left(\int_{x_0}^x H(x, t)f_1^p(x)dx \right)^{\frac{-1}{p}} \times \left(\int_{x_0}^x H(x, t)f_2^q(x)dx \right)^{\frac{-1}{q}} \right\}.$$

5. Generalization of previous results

In this last section, we want to give improvements of the previous results.

THEOREM 5.1 Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be measure spaces with σ -finite measures and $f_i: \Omega_2 \rightarrow \mathbb{R}, i = 1, \dots, 2n$, be non-negative functions. Let g be the function that has the following representation

$$g(x) = \int_{\Omega_1} k(x, t)f(t)d\mu_1(t) \tag{5.1}$$

where $k: \Omega_2 \times \Omega_1 \rightarrow \mathbb{R}$ is a general non-negative kernel and $f: \Omega_1 \rightarrow \mathbb{R}$ be a real-valued function. If $p_i, i = 1, \dots, n$ are positive real numbers such that $\sum_{i=1}^n \frac{1}{p_i} = 1$, then

$$\int_{\Omega_2} \prod_{j=1}^n f_j(x)g(x)d\mu_2(x) \leq C \prod_{j=1}^n \left(\int_{\Omega_2} f_{n+j}(x)g(x)d\mu_2(x) \right)^{\frac{1}{p_j}} \tag{5.2}$$

where

$$C = \sup_{t \in \Omega_1} \left\{ \left(\int_{\Omega_2} k(x, t)f_1(x) \dots f_n(x)d\mu_2(x) \right) \prod_{j=1}^n \left(\int_{\Omega_2} k(x, t)f_{n+j}(x)d\mu_2(x) \right)^{\frac{-1}{p_j}} \right\}.$$

Proof For fixed $\lambda > 0, \lambda \neq 1$ we define c_i as follows

$$c_i = \frac{1}{\ln \lambda} \left(\ln \frac{\prod_{j=1}^n \left(\int_{\Omega_2} f_{n+j}(x)g(x)d\mu_2(x) \right)^{\frac{1}{p_j}}}{\int_{\Omega_2} f_{n+i}(x)g(x)d\mu_2(x)} \right), i = 1, \dots, n.$$

For these numbers, one can observe

$$\sum_{i=1}^n \frac{c_i}{p_i} = 0$$

and from this we obtain

$$\prod_{i=1}^n (\lambda^{c_i})^{\frac{1}{p_i}} = 1. \tag{5.3}$$

Using the representation of g , and Equation 5.3, we have

$$\begin{aligned} \int_{\Omega_2} f_1(x) \dots f_n(x)g(x)d\mu_2(x) &= \int_{\Omega_2} \int_{\Omega_1} f_1(x) \dots f_n(x)k(x, t)f(t)d\mu_2(x)d\mu_1(t) \\ &= \int_{\Omega_1} \left(\int_{\Omega_2} k(x, t)f_1(x) \dots f_n(x)d\mu_2(x) \right) f(t)d\mu_1(t) \\ &\leq C \int_{\Omega_1} \prod_{j=1}^n \left(\int_{\Omega_2} f_{n+j}(x)k(x, t)d\mu_2(x) \right)^{\frac{1}{p_j}} f(t)d\mu_1(t) \\ &= C \int_{\Omega_1} \prod_{j=1}^n \left(\int_{\Omega_2} \lambda^{c_j} f_{n+j}(x)k(x, t)d\mu_2(x) \right)^{\frac{1}{p_j}} f(t)d\mu_1(t) \end{aligned}$$

Now, using the inequality between arithmetic and geometric means, we have

$$\leq C \int_{\Omega_1} \sum_{j=1}^n \frac{\int_{\Omega_2} \lambda^c f_{n+j}(x) k(x, t) d\mu_2(x)}{p_j} f(t) d\mu_1(t).$$

Thus, we have

$$\int_{\Omega_2} f_1(x) \dots f_n(x) g(x) d\mu_2(x) \leq C \sum_{j=1}^n \frac{\lambda^c}{p_j} \int_{\Omega_2} f_{n+j}(x) g(x) d\mu_2(x).$$

Using definition of c_p , we get (Equation 5.2). □

COROLLARY 5.2 If we set $f_{n+j}(x) = f_j(x)^{p_j}$, then we have

$$\int_{\Omega_2} \prod_{j=1}^n f_j(x) g(x) d\mu_2(x) \leq C \prod_{j=1}^n \left(\int_{\Omega_2} f_j(x)^{p_j} g(x) d\mu_2(x) \right)^{\frac{1}{p_j}}$$

where

$$C = \sup_{t \in \Omega_1} \left\{ \left(\int_{\Omega_2} k(x, t) f_1(x) \dots f_n(x) d\mu_2(x) \right) \prod_{j=1}^n \left(\int_{\Omega_2} k(x, t) f_j(x)^{p_j} d\mu_2(x) \right)^{-\frac{1}{p_j}} \right\}.$$

COROLLARY 5.3 If we put $f(t) = 1$ and $\Omega_1 = \Omega_2 = [0, \infty]$ and replace $d\mu_1(t)$, $d\mu_2(x)$ by measures $dh(t)$, dx , respectively, in Theorem 5.1, then we get Varošeneć (1995, Theorem 1). Further, if we set $f_{n+j}(x) = f_j(x)^{p_j}$, we get a generalization of Volkov's result (1972).

Remark 5.4 If we take $n = 2$ in Equation 5.2 of Theorem 5.1, we get Equation 2.2 of Theorem 2.

Remark 5.5 Applications of results in this section for various types of fractional integrals and fractional derivatives can be observed in Sections 3 and 4 and we omit the details.

Funding

The authors received no direct funding for this research.

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Citation information

Cite this article as: On generalization of an integral inequality and its applications, Ghulam Farid, Sajid Iqbal & Josip Pečarić, *Cogent Mathematics* (2015), 2: 1066528.

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