Matrix $\ell$-algebras over $\ell$-fields

Jingjing Ma1*

Abstract: It is shown that if a matrix $\ell$-algebra $M_n(K)$ over certain $\ell$-fields $K$ contains a positive $n$-cycle $e$ such that $I + e + \cdots + e^{n-1}$ is a $d$-element on $K$ then it is isomorphic to the $\ell$-algebra $M_n(K)$ over $K$ with the entrywise lattice order.

Subjects: Advanced Mathematics; Algebra; Fields & Rings; Mathematics & Statistics; Science

Keywords: $d$-element; lattice-ordered algebra; lattice-ordered field; matrix algebra; positive cycle

AMS subject classification: 06F25

1. Introduction

Let $R$ be a lattice-ordered ring ($\ell$-ring) with the positive identity element and $M_n(R)$ ($n \geq 2$) be the $n \times n$ matrix ring over $R$. $M_n(R)$ may be made into an $\ell$-ring by defining a matrix in $M_n(R)$ positive if each entry of the matrix is positive in $R$. This lattice order on $M_n(R)$ is called the entrywise lattice order. Since $R$ has the positive identity element, the identity matrix of $M_n(R)$ is positive with respect to the entrywise lattice order. In 1966, Weinberg first proved that if $M_2(Q)$ is an $\ell$-ring in which the identity matrix is positive, where $Q$ is the field of rational numbers, then it is isomorphic to the $\ell$-ring $M_2(Q)$ with the entrywise lattice order (Weinberg, 1966). Then in Ma and Wojciechowski (2002) it was proven that this fact is true for any totally ordered subfield $F$ of the field $\mathbb{R}$ of real numbers and any $n \times n$ matrix algebra over $F$. Whether or not the above fact is true for an $n \times n$ ($n \geq 2$) matrix algebra over an arbitrary totally ordered field is still an open question. Under some stronger conditions, though, the result is true; for instance, if an $\ell$-algebra $M_n(I)$ ($n \geq 2$) contains a positive $n$-cycle, where $T$ is an arbitrary totally ordered field, then it is isomorphic to the $\ell$-algebra $M_n(I)$ with the

ABOUT THE AUTHOR

Jingjing Ma’s research is in the area of Lattice-ordered Rings and Algebras that was first systematically studied by G Birkhoff and RS Pierce about 60 years ago. His interests involve different topics on lattice-ordered rings with the focus on trying to better understand the algebraic structures of such systems.

The algebraic structure of lattice-ordered matrix algebras has been one of his research topics. In 2000, working together with Wojciechowski (University of Texas at El Paso), they proved the so-called Weinberg’s conjecture, which states that there is only one lattice order on matrix rings over the field of rational number in which the identity matrix is positive. The present work continues the study of lattice-ordered matrix algebras in a more general setting. More precisely, this article considers lattice-ordered matrix algebras over certain lattice-ordered fields, which include totally ordered fields as a special case.

PUBLIC INTEREST STATEMENT

Given a positive integer $n$, a $n$ by $n$ matrix is a square array consisting of $n$ rows and $n$ columns of numbers. We could add two matrices, multiply two matrices, and compare two matrices by certain rules. Matrices with those operations consist of a very important research area in mathematics, and it has rich applications to other areas in mathematics, physics, etc.

The present research continues the study of matrices with the operations and order. It considers matrices with the operations and order in a more general setting to generalize some important known results. The article exposes more properties of such matrices to better understand them.
entrywise lattice order \cite{Ma2000}. The reader is referred to Steinberg \cite{Steinberg2010} for more information on research activities in this area.

The same problem for matrices over a lattice-ordered field (\(\ell\)-field) has been not studied. In this article, we consider \(n \times n\) matrix \(\ell\)-algebra over certain Archimedean \(\ell\)-fields. We first provide an example to show that the \(n \times n\) matrix \(\ell\)-algebra \(\mathcal{M}_n(K)\) over an \(\ell\)-field \(K\) may not be isomorphic to the \(\mathcal{M}_n(K)\) with entrywise lattice order even it contains a positive \(n\)-cycle.

Example 1.1 Consider the \(\ell\)-field \(K = \mathbb{Q}([\sqrt{2}]) = \{a + \beta\sqrt{2} \mid a, \beta \in \mathbb{Q}\} \subseteq \mathbb{R}\) with coordinatewise lattice order, that is, \(a + \beta\sqrt{2} \geq 0\) if \(a \geq 0\) and \(\beta \geq 0\) in \(\mathbb{Q}\). Now define a matrix \(a = (a_{ij}) \in \mathcal{M}_4(K)\) to be positive if each \(a_{ij}\) is greater than or equal to zero with respect to the usual total order in \(\mathbb{R}\). Since the coordinatewise lattice order in \(K\) contains in the total order induced from \(\mathbb{R}\), \(\mathcal{M}_n(K)\) is an \(\ell\)-algebra over \(K\) and the identity matrix is positive under this order. We notice that this lattice order is not the entrywise lattice order on \(\mathcal{M}_n(K)\) over the \(\ell\)-field \(K\).\footnote{In the following, we always assume that \(I\) is the identity matrix, but \(I = \mathbb{Q}\) is not positive in \(K\) with respect to the coordinatewise lattice order. The \(n \times n\) matrix \(e = e_{12} + e_{23} + \cdots + e_{n-1,n} + e_{nn}\) is an \(n\)-cycle as defined below, where \(e_{ij}\) are standard matrix units. It is clear that \(e\) is positive in the above lattice order defined on \(\mathcal{M}_n(K)\).

The main result of this article is a proof that for a certain \(\ell\)-field \(K\), if the \(\ell\)-algebra \(\mathcal{M}_n(K)\) over \(K\) possesses a positive \(n\)-cycle \(e\) such that \(I + e + \cdots + e^{n-1}\) is also a \(d\)-element on \(K\) then the \(\ell\)-algebra is isomorphic to the \(\ell\)-algebra \(\mathcal{M}_n(K)\) with the entrywise lattice order. We will briefly review few definitions and results, the reader is referred to Birkhoff and Pierce \cite{Birkhoff1956} and Steinberg \cite{Steinberg2010} for general information on \(\ell\)-rings and undefined terminologies.

We call the permutation matrix \(e_{i,j} + \cdots + e_{i,k-1} + e_{i,k}\) as an \(n\)-cycle in the \(n \times n\) matrix ring, where \(e_{ij}\) are the standard matrix units, that is, \(i\)th entry of \(e_{ij}\) is 1, and other entries are zero. For an \(\ell\)-field \(K\), the \(\ell\)-ring \(\mathcal{M}_n(K)\) is called a lattice-ordered algebra (\(\ell\)-algebra) if for any \(a \in K\) and \(a \in \mathcal{M}_n(K)\), \(a \geq 0\) if \(a \geq 0\). A positive element \(x\) of \(\mathcal{M}_n(K)\) is called a \(d\)-element on \(\mathcal{M}_n(K)\) (on \(K\)) if for any \(a, b \in \mathcal{M}_n(K)\), \(a \wedge b = 0\) \(\Rightarrow\) \(ax \wedge xb = 0\) in \(\mathcal{M}_n(K)\). If for any \(u, v \in K\), \(u \wedge v = 0\) \(\Rightarrow\) \(ux \wedge vx = 0\) in \(\mathcal{M}_n(K)\), then \(a \wedge b = 0\) \(\Rightarrow\) \(ax \wedge bx = 0\) in \(\mathcal{M}_n(K)\). It is well known that if \(u, u^{-1} \in K\) are both positive then \(u\) is a \(d\)-element both on \(\mathcal{M}_n(K)\) and on \(K\), and if \(a, a^{-1} \in \mathcal{M}_n(K)\) are both positive, then \(a\) is a \(d\)-element on \(\mathcal{M}_n(K)\).

2. Main result
Let \(K\) be an \(\ell\)-field with a positive identity element 1. It is well known that 1 is a basic element in the sense that for any \(a, b \in K^+\), \(a \geq b \) or \(b \geq a\). Define \(F = \{x \in K \mid |x|\text{ is an }\ell\text{-element of }K\}\). Then \(F\) is the largest totally ordered subfield of \(K\) \cite{Schwartz1986, Theorem 4}.\footnote{In the following, we always assume that \(K\) is an Archimedean \(\ell\)-field and finite-dimensional as a vector space over \(F\). Then \(K\) has a \(\sqrt{\ell}\)-basis \(\{u_1, \ldots, u_m\}\) \cite{Schwartz1986, Corollary, p. 186}, that is, \(K = Fu_1 + \cdots + Fu_m\) and \(a_1u_1 + \cdots + a_mu_m \geq 0\), where \(a_i \in F\), if and only if each \(a_i \geq 0\). Moreover, we assume that each \(u_j\) is a \(d\)-element on \(K\). Then \(u_j^{-1} > 0\) since \(1 = |1| = |u_j^{-1}u_j| = |u_j^{-1}|u_j\) implies that \(u_j^{-1} > 0\).}

A simple example of this situation is \(K = \mathbb{Q}([\sqrt{2}]) = \{a + \beta \sqrt{2} \mid a, \beta \in \mathbb{Q}\}\) with the coordinatewise lattice order. Then \(u_1 = 1, u_2 = \sqrt{2}\) is a \(\sqrt{\ell}\)-basis of \(K\) over \(\mathbb{Q}\). It is clear that \(F = \mathbb{Q}\) and \(u_1, u_2\) are \(d\)-elements on \(K\) and on any \(\ell\)-algebra \(\mathcal{M}_n(K)\).

Suppose that \(K\) is an Archimedean \(\ell\)-field with \(1 > 0\) and suppose \(K\) is finite-dimensional over \(F\) with a \(\sqrt{\ell}\)-basis that consists of \(d\)-elements of \(K\). Consider the \(n \times n\) matrix algebra \(A = \mathcal{M}_n(K)\) over \(K\) with \(n \geq 2\). Suppose that \(A\) is an \(\ell\)-algebra over \(K\). Then \(A\) is also an \(\ell\)-algebra over \(F\) since the
lattice order on $K$ extends the total order of $F$. Since $A$ contains no nonzero nilpotent ideals, $A$ is Archimedean over $F$ (Birkhoff & Pierce, 1956, Corollary 1, p. 51). Since $A$ is also finite-dimensional over $F$, it implies that $A$ is a direct sum of maximal convex totally ordered subspaces over $F$ (Conrad, 1961, Theorem 7.3, p. 232). Let $(u_1, \ldots, u_m)$ be a $\varepsilon$-basis of $K$ over $F$ in which each $u_i$ is a $d$-element of $K$. We will assume that $u_1 = 1$, the identity element of $K$. The identity matrix of $M_n(K)$ is denoted by $I$. The following is the main result of the paper.

**Theorem 2.1** Suppose that $A = M_n(K)$ is an \(\varepsilon\)-algebra over $K$. If $A$ contains a positive $n$-cycle $e$ and $I + e + \cdots + e^{n-1}$ is a $d$-element on $K$ then the $\varepsilon$-algebra $A$ is isomorphic to the $\varepsilon$-algebra $M_n(K)$ over $K$ with the entrywise lattice order.

**Proof** As we discussed in the previous paragraph, $A$ is a direct sum of maximal convex totally ordered subspaces of $A$ over $F$. Since the identity matrix $I = e^n > 0$, $I$ is a sum of disjoint basic elements. Suppose that $0 < a \leq 1$ is a basic element. Since $u_i^{a^n} > 0$ for each $u_i$, every $u_i$ is also a basic element of $A$. For $k = 1, \ldots, m$, define $M_k = (u_i a)^{a^n}$ and $H_k = \bigcup_{j=1}^m e^j M_k e^j$. Then each $M_k$ is a maximal convex totally ordered subspace of $A$ over $F$ since $A$ is a $f$-module over $F$, and each $H_k$ is a convex $\varepsilon$-subspace of $A$ over $F$. We divide the proof of Theorem 2.1 into several lemmas.

**Lemma 2.2** $H_i \cap H_j = \{0\}$ for $1 \leq i, j \leq m$ and $i \neq j$.

**Proof** Clearly the sets $\{e^j M_k e^j | 1 \leq j \leq n\}$ and $\{e^j M_k e^j | 1 \leq j < n\}$ of maximal convex totally ordered subspaces over $F$ are either identical or disjoint (Conrad, 1961, Lemma 3.1). Thus either $H_i = H_j$ or $H_i \cap H_j = \{0\}$. Suppose that $H_i \cap H_j \neq \{0\}$. Then $H_i = H_j$, so $u_i a \in H_i$ and hence $u_i a \in e^j M_k e^j$ for some $1 \leq j < n$ since $u_i a$ is basic. Thus, $u_i a$ and $e^j (u_i a) e^j$ are comparable. On the other hand, $u_i a \leq u_i I$ and $I + e + \cdots + e^{n-1}$ is a $d$-element on $K$, so

$$(u_i (I + e + \cdots + e^{n-1}) \wedge u_i (I + e + \cdots + e^{n-1})) = 0.$$  

Since $u_i a \leq u_i I$, 

$$(e^j (u_i a) e^j \leq e^j (u_i a) e^{j+1} \leq u_i (I + e + \cdots + e^{n-1}),$$  

and also $u_i a \leq u_i I \leq u_i (I + e + \cdots + e^{n-1})$. Thus $e^j (u_i a) e^j \wedge u_i a = 0$, which is a contradiction. Therefore, we must have $H_i \cap H_j = \{0\}$ for $i \neq j$.

**Lemma 2.3** $A = H_1 + \cdots + H_m$.

**Proof** Let $f_k = \sum_{j=1}^m e^j (u_i a) e^j \in H_k$, $k = 1, \ldots, m$. Then by Lemma 2.2, $\{f_1, \ldots, f_m\}$ is a disjoint set, that is, $f_i \wedge f_j = 0, i \neq j$. For each $f_k$, since $ef_k = f_k e = f_k$, we have

$$f_k = u_i (I + e + \cdots + e^{n-1}), \quad 0 \neq v_k \in K, \quad k = 1, \ldots, m.$$  

Indeed, since $e^n = I$ and $e^0 = I$ for $1 \leq m < n$, the characteristic polynomial of $e$ is $x^n - 1$, so $e$ has $n$ distinct eigenvalues and hence the eigenspace for each eigenvalue is of one-dimensional. From $ef_k = f_k e$, each column in $f_k$ is an eigenvector of $e$ to 1. Then each column in $f_k$ is a scalar multiple of the vector $v_k$, each of whose components is equal to 1. Similarly, each row of $f_k$ is a scalar multiple of $v_k^t$, the transpose of $v$. Thus, $f_k = v_k (I + e + \cdots + e^{n-1})$ for some $0 \neq v_k \in K$.

Let $v_k \wedge 0 = w_k$ in $K$. Then since $I + e + \cdots + e^{n-1}$ is a $d$-element on $K$,

$$0 = f_k \wedge 0 = u_i (I + e + \cdots + e^{n-1}) \wedge u_i (I + e + \cdots + e^{n-1}),$$  

so $w_k = 0$, and hence $v_k > 0$ for each $k$. Similarly we show that $v_k \wedge v_k = 0$ for $k \neq k$, and so $\{v_1, \ldots, v_m\}$ is a disjoint set in $K$. Since each positive element in $K$ is a positive linear combination of $u_1, \ldots, u_m$, each
\(v_i\) is a strictly positive multiple by a scalar in \(F\) of exactly one of \(u_1, \ldots, u_m\), so without loss of generality, we may assume that \(v_i = a_i u_p\) where \(0 < a_i \in F, i = 1, \ldots, m\).

Suppose that \(A \neq H_1 + \ldots + H_m\). Then there exists a maximal convex totally ordered subspace \(M\) of \(A\) over \(F\) which is not in the sum of \(H_1 + \ldots + H_m\). Let \(0 < x \in M\) and \(g = \sum_{j=1}^n e^j x^j\). Then \(eg = ge = g\) and by a similar argument as before, \(g = v(I + e + \ldots + e^{n-1})\) for some \(0 < v \in K\), and \(v \wedge v_i = 0\) in \(K\) for \(i = 1, \ldots, m\). Hence \(v \wedge u_j = 0\) in \(K\) for \(i = 1, \ldots, m\), so \(v = 0\), which is a contradiction. Therefore, \(A = H_1 + \ldots + H_m\).

**Lemma 2.4** For each \(k = 1, \ldots, m\), \(H_k = \sum_{i=1}^n e^i M_k e^i\) is a direct sum.

**Proof** To prove that \(H_k = \sum_{i=1}^n e^i M_k e^i\) is a direct sum, we show that any two summands are different, and thus they must have zero intersection (Conrad, 1961). For \(1 \leq s, t \leq n\) with \(s \neq n\) or \(t \neq n\), suppose first that \(e^i M_k e^i = M_k \neq 0\). Then for any \(0 \leq x \in M_k\), \(e^i x^j = x\) and \(x\) is comparable. If \(x > e^i x^j\) then \(x > e^i x^j > e^2 x^j \ldots > e^m x^j = x\), which is a contradiction. Similarly, \(x < e^i x^j\). Thus, \(e^i x^j = x\) for all \(x \in M_k\), so \(e^i x^j = e^j x^j = g\) for all \(g \in H_f\). For any \(y \in M_k\), \(e^j y^j = 0\) since \(y, u_j\) are comparable, \(u_j y^j = y^j u_j = y\) by \(e^j e^j = e^j M_k e^j\) and previous arguments. Hence, \(e^i y^j = y\) for all \(y \in M_k\), so \(e^i h^j = h\) for all \(h \in H_k\), \(e^j \neq k\). Since \(A = H_1 + \ldots + H_m\) by Lemma 2.3, we have \(e_i f^i = f\) for all \(f \in A\). In particular, \(e^j i = I\), so \(s + t = n\), and hence \(s \neq n\) and \(t \neq n\). Therefore, \(e_i f = e_i f^j = e_i e^j e^j e^i = f^i\) for any \(f \in A\), that is, \(e_i\) is in the center of \(A\) with \(1 \leq s < n\), which is a contradiction. Hence, for \(1 \leq s, t \leq n\) with \(s \neq n\) or \(t \neq n\), \(e_i M_k e^i \neq M_k\).

Now for \(1 \leq i_1, i_2, j_1, j_2 \leq n\), suppose that \(e^{i_1} M_k e^{i_1} = e^{i_2} M_k e^{i_2}\). If \(i_1 < i_2\) then \(M_k = e^{i_1} M_k e^{i_2}\) if \(j_2 \geq j_1\), which is a contradiction by previous paragraph. If \(j_1 < j_2\) then \(M_k = e^{j_1} M_k e^{j_2}\), which is again a contradiction. Thus, \(i_1 \neq i_2\). Similarly, \(i_2 \neq i_3\), and hence \(i_1 \neq i_3\). Similarly, \(j_1 \neq j_2\). Therefore, \(H_k = \sum_{i=1}^n e^i M_k e^i\) is a direct sum of \(n\) maximal convex totally ordered subspaces \(e^i M_k e^i\), \(1 \leq i, j \leq n\).

We notice that since \(\dim F = mn\) and \(A\) is a direct sum of \(mn\) totally ordered subspaces over \(F\), each \(M_i, i = 1, \ldots, m\), is one-dimensional over \(F\), so \(M_i = F(u_i a)\) for each \(i = 1, \ldots, m\).

**Lemma 2.5** \(H_1\) is isomorphic to the \(e\)-algebra \(M_i(F)\) over \(F\) with the entrywise lattice order.

**Proof** First, we notice that \(I = a \in M_1\), so \(e = e I = I e\) implies that \(e M_1 = M_1 e\), which is a contradiction by Lemma 2.5. Suppose that \(I = a + a_1 + \ldots + a_p\) where \(a, a_1, \ldots, a_p\) are disjoint basic elements and \(p \geq 1\). For each \(i = 1, \ldots, n - 1\), \(I = e^i e^{i+1} + \ldots + e^p e^{p-1}\) implies that each \(e^j e^{i-1}\) is an \(f\)-element for \(i = 1, \ldots, n\), and \(e^j e^{i-1}\) is also a basic element since \(e^j = I\).

We claim that \(I = a + e a e^{-1} + \ldots + e^{i-1} a e\). For \(i = 1, \ldots, n - 1\), since \(e^j e^{i-1} < I\) is basic, there exists \(a_j\) for some \(j = 1, \ldots, p\) such that \(e^j e^{i-1}\) and \(a_j\) are comparable. We show that \(e^j e^{i-1} = a_j\). Since \(e^j e^i = a_j\) and \(a_j\) are comparable, \(e^j e^i \wedge a_j = 0\) for any \(1 \leq v \leq p\) and \(v \neq j\). Otherwise \(e^j e^{i-1}\) and \(a_j\) will be comparable since both are basic elements, so \(a_j \wedge e^i e^{i+1} = 0\) for all \(i = 1, \ldots, n - 1\), which implies that \(a_j\) and \(a_j\) are comparable, a contradiction. Then from \(a + a_1 + \ldots + a_p = I = e^i e^j e^{i-1} + \ldots + e^p e^{p-1}\),

we have \(e^i e^{i-1} \leq a_j\) and similarly \(a_j \leq e^i e^{i-1}\). Thus \(e^i e^{i-1} = a_j\) so each \(e^i e^{i-1} = a_j\), \(i = 1, \ldots, n - 1\), appears in the sum \(I = a + a_1 + \ldots + a_p\).

Next, we show that each \(a_j, 1 \leq r \leq p\), is equal to \(e^i e^{i-1}\) for some \(1 \leq i \leq n - 1\). We first notice that since \(e^i e^{i-1} = i = 0, \ldots, n - 1\), appear in the sum for \(I\), they are disjoint \(f\)-element, so \(ae^i = 0\) for \(i = 1, \ldots, n - 1\). Since \(a_j\) is a basic vector, \(a_j = e^i e^j\) for some \(0 < x \in M_w\) and \(1 \leq s, t \leq n, 1 \leq w \leq m\). Then that \(a_j\) is idempotent implies that \(e^i e^j e^i e^j = e^j x^j\), so \(e^j x = x\). Suppose that \(s + t = n + v\), where \(0 \leq v < n\). Thus \(x^j x = x\). From \(M_w = t(u_w a)\), we have \(x = a(u_w a)\) for some
$0 < a \in F$, then $au_{a}(ae \cdot a) = a$, and hence $v = 0$, otherwise $ae \cdot a = 0$. It follows from $au_{a} = a$ that $au_{a} = 1$, an identity element of $K$. Therefore, $a_{r} = e^{r}ae^{-r}$.

Since $I = a + eae^{-1} + \cdots + e^{n-1}ae \in H_{1}$ is a sum of disjoint elements, each $e^{r}ae^{-1}$ is an idempotent $f$-element and $e^{r}ae^{-1}e^{r}ae^{-1} = 0$ for $i \neq j$. Thus, $H_{1} = \sum_{j=1}^{n} F(e^{r}ae^{-1})$ is an $e$-algebra over $F$. For $1 \leq i,j \leq n$, define $c_{ij} = e^{r}ae^{-1}$. Then $\{c_{ij} \mid 1 \leq i,j \leq n\}$ is a disjoint set of basic elements and also a set of $n \times n$ matrix units. It follows that $H_{1} = \sum_{j=1}^{n} F_{c_{ij}}$ is a direct sum as a vector lattice over $F$. Define $\varphi: H_{1} \rightarrow M_{n}(F)$ by $\sum_{j=1}^{n} a_{ij} e_{ij} \rightarrow \sum_{j=1}^{n} a_{ij} e_{ij}$, where $a_{ij} \in F$. Then $\varphi$ is an $e$-isomorphism between two $e$-algebras over $F$, so $H_{1}$ is isomorphic to the $e$-algebra $M_{n}(K)$ over $K$ with the entrywise lattice order. This completes the proof of Theorem 2.1.

Example 1 shows that the condition “$I + e + \cdots + e^{n-1}$ is a $d$-element on $K$” couldn’t be omitted in Theorem 2.1. However, if $K$ is a totally ordered field then this condition is automatically satisfied, so it is not necessary.

By Schwartz (1986, Theorem 10), under the assumption for the $e$-field $K$ in Theorem 2.1, the lattice order on $K$ is uniquely extensible to a total order. Although we believe that if $I + e + \cdots + e^{n-1}$ is not a $d$-element on $K$ then the lattice order on $M_{n}(K)$ is defined by the unique total order on $K$ just like the situation in Example 1.1, we lack the ability to prove it. In the next, we will show that this fact is true for the simplest case.

### 3.2 $2 \times 2$ case

Let $K$ be an Archimedean $e$-field which is a two-dimensional extension of $F$ and let $(I, u)$ be a $v$-basis of $K$ over $F$ and let $u$ be a $d$-element. Suppose that $M_{2}(K)$ is a $2 \times 2$ matrix $e$-algebra over $K$ and that $e = e_{12} + e_{21} \geq 0$. By Theorem 2.1, we know that if $I + e$ is a $d$-element on $K$ then $M_{2}(K)$ is isomorphic to the $e$-algebra $M_{2}(K)$ with the entrywise lattice order. In this section, we show that if $I + e$ is not a $d$-element on $K$ then the lattice order on $M_{2}(K)$ is the lattice order defined in Example 1.1.

**Theorem 3.1** Let $M_{2}(K)$ be an $e$-algebra over $K$ in which $e = e_{12} + e_{21}$ is positive, but $I + e$ is not a $d$-element on $K$. Then the lattice order on $M_{2}(K)$ is defined by declaring a matrix $a = (a_{ij}) \in M_{2}(K)$ to be positive if each $a_{ij} \geq 0$ in $K$, where $\geq$ is the unique total order that extends the lattice order on $K$.

**Proof** As before, suppose that $F$ is the unique largest totally ordered subfield of $K$ and $(I, u)$ is an $v$-basis of $K$ over $F$ where $u$ is a $d$-element. From the discussion in the previous section, $M_{2}(K)$ is a direct sum of totally ordered subspaces of $K$ over $F$.

We first show that $I \land e = 0$. Let $I \land e = Z$. Then $eZ = e = e \land e^{2} = e \land I = Z$, so $Z = v(I + e)$ for some $v \in K$. Since $0 \leq z \leq I$, $z$ is an $f$-element of $M_{2}(K)$, so $I \land e = Z$ implies that $z^{2} = z \land ze = z \land z = Z$. Thus, $Z = v(I + e)$ implies that $2v^{2} = Z$, and hence either $v = 0$ or $2v = 1$. If $2v = 1$, then $I + e = 2Z \leq 2I$ implies that $e \leq I$, which is a contradiction. Thus, we must have $v = 0$, so $I \land e = Z = 0$.

Next, we show that $I$ cannot be a basic element. Assume, on the contrary, that $I$ is basic. Therefore, $e$ is also basic. Suppose that $M_{2}(K) = T_{1} \oplus \cdots \oplus T_{k}$, where each $T_{k}$ is a totally ordered subspace of $M_{2}(K)$ over $F$ and $k \geq 2$. We may assume that $I \in T_{1}$ and $e \in T_{2}$. We consider two cases.

1. $k = 2$. Take $0 < a \in T_{2}$. Then $e \land a = 0$ implies that $I \land ea = I \land a = 0$, so $ea \in T_{2}$. Hence, $ea$ and $ea$ are comparable since $T_{2}$ is totally ordered. If $ae < ea$ then $a = ae^{2} < (ea)e < e(ea) = a$, which is a contradiction. Similarly, $ea < ea$. Thus, we must have $ea = ea$, so for any element $x \in T_{1}$, $ex = xe$. Similarly for each $y \in T_{2}$, $ey = ye$. Therefore, for each matrix $w \in M_{2}(K)$, $ew = we$, which is a contradiction.
(2) $k \geq 3$. Define $H = \sum_{i,j=1}^{2} e^i T^j e^j$ and $N = \sum_{i,j=1}^{2} e^i T^j e^j$. Then $H = T_1 \oplus T_2$. Since $I, e$ are basic elements, it is straightforward to check that $u_I, u_e$ are also basic elements. If $u_I \in T_1$ then $u(I + e) \in N$. Since $H \neq N, H \cap N = \{0\}$. On the other hand, since $I + e$ is not a $d$-element on $K$, $0 \neq (I + e) \cap u(I + e) \in H \cap N$, which is a contradiction. Thus, we must have $u_I \in H$, and hence $u(I + e) \in H$. Since each element in $K$ can be written as $a + \beta u$ with $a, \beta \in F$, that $I + e$ and $u(I + e)$ are in $H$ implies that $v(I + e) \in H$ for any $v \in K$. Take $0 < a \in T_1$ and form $b = \sum_{i,j=1}^{2} e^i a e^j \in N$. Then $be = eb = b$ implies that $b = v(I + e)$ for some $0 < v \in K$. Then $b \in H$, which contradicts the fact that $H \cap N = \{0\}$. Hence, $I$ cannot be a basic element.

Since $I$ is not basic, $I = a_1 + \cdots + a_m$ where $a_1, \ldots, a_m$ are disjoint basic elements with $2 \leq m \leq 4$ since $e a_1, \ldots, e a_m$ are also disjoint and $M_2(K)$ is eight-dimensional over $F$. Since $a_i \leq I$, each $a_i$ is an $f$-element, and hence $a_i \wedge a_i = 0$ for $i \neq j$ implies that $a_i a_j = a_i \wedge a_j = 0$. Thus, each $a_i$ is idempotent. From $I = a_1 + \cdots + a_m$, we have $I = e a_1 + \cdots + e a_m$ with $e a_i e = 0$ for $i \neq j$ since $e$ is a $d$-element on $M_2(K)$. Thus, each $e a_i e = a_i$ for some $t$. If $s = t$ then $e a_i = a_i e$ and since $a_i$ is idempotent, $a_i = \frac{1}{2}(I + e)$, which is a contradiction since $a_i$ is basic.

We claim that $m = 2$. If $m = 3$ then we have $I = a_1 + a_2 + a_3 = e a_1 e + e a_2 e + e a_3 e$. Since $e a_1 e \neq e a_2$, we may assume $e a_1 e = a_2$. Then $e a_1 e = a_2$ implies that $e a_1 e = a_2$ is a contradiction. Suppose that $m = 4$. Then $(a_1, a_2, a_3, a_4, e a_1, e a_2, e a_3, e a_4)$ is disjoint and $M_2(K)$ is eight-dimensional over $F$, so $\{a_1, a_2, a_3, e a_1, e a_2, e a_3, e a_4\}$ is a $v^r$-basis of $M_2(K)$ over $F$. It is straightforward to check that $ua_1$ is a basic element, and hence $u a_1 = a_1 e$ or $u a_1 = e a_1$ for some $0 < a, \beta \in F$. In the first case, we have $u = a$, which is a contradiction. So $u a_1 = e a_1$, and hence $\beta^2 a_1 = \beta (e a_1) = \beta (e a_1) = u^2 a_1$. Hence $u^2 = \beta^2$, so $u = \beta$ or $-\beta$, which is again a contradiction. Therefore, we must have $m \neq 4$, so $m = 2$.

Thus, we have $I = a_1 + a_2$ where $a_1, a_2$ are disjoint basic elements. Then as we have discussed before $e a_1 e = a_2$ and $I = a_1 + e a_2 e$. Let $M_2(K) = T_1 \oplus \cdots \oplus T_v$, where $k \geq 4$ and each $T_i$ is a maximal convex totally ordered subspace of $M_2(K)$ over $F$. Assume that $a_i \in T_i$, and define $H = \sum_{i,j=1}^{2} e^i T^j e^j = T_1 + \cdots + T_v + e T_1 + e T_v$. Then $I + e \in H$. Since $u a_1$ is basic, $u a_1 \in T_i$ for some $T_i$. If $T_i$ is not in $\{T_1, T_v, e T_1, e T_v\}$ then $H \cap N = \{0\}$, where $N = \sum_{i,j=1}^{2} e^i T^j e^j$. On the other hand, $u a_1 \in T_i$ implies that $u(I + e) \in N$, then $0 \neq (I + e) \cap u(I + e) \in H \cap N$, which is a contradiction. Thus, $T_i$ is one of $T_1, e T_1, T_v, e T_v$, so $u a_1$ is in one of $\{T_1, e T_1, T_v, e T_v\}$. If $u a_1 \leq e a_2$, then $u a_1 = u a_2 \leq a_1 e a_1 = 0$, a contradiction. If $e a_2 \leq u a_1$, then $a_1 \leq u a_2$, so $a_1 = a_1^2 \leq u a_2 e a_1 = 0$. Similarly $u a_1$ is not comparable with $a_1$ and $e a_2$, so $u a_1 \in T_i$. Since $a_1$ and $u a_1$ are linearly independent over $F$, $T_i$ is two-dimensional over $F$, and hence $e T_1, e T_v$ are all two-dimensional over $F$. Hence, $M_2(K) = T_1 \oplus e T_1 \oplus T_v \oplus e T_v$ as a vector lattice over $F$.

Let us consider the structure of $T_i$ first. Since $a_1, u a_1$ are linearly independent over $F$, $T_i = \langle a_1, u a_1 \mid a_1, u a_1 \beta \in F \rangle = Ka_1$. Then clearly $T_i$ is a field that is isomorphic to $K$ under the mapping $a_1 \rightarrow v$ for any $v \in K$. Since $T_i$ is totally ordered, $K$ will be totally ordered if we define an order $\geq$ on $K$ by saying that for any $v \in K, v \geq 0$ if $u a_1 \geq 0$ in $T_i$. Take $v \in K$, if $v \geq 0$ in $K$ then since $M_2(K)$ is an $r$-algebra over $K$, we have $u a_1 \geq 0$ in $M_2(K)$, so $v \geq 0$ in $K$. Thus, the total order $\geq$ on $K$ extends the lattice order $\geq$ on $K$, and hence $u a_1 \geq 0$ in $T_i$. Hence, for any $v \in K, v \geq 0$ if and only if $u a_1 \geq 0$ in $M_2(K)$.

Since $M_2(K) = T_1 \oplus T_v \oplus e T_1 \oplus e T_v e = Ka_1 \oplus K(a_1 e) \oplus K(e a_1) \oplus K(e a_1 e)$, if we define the mapping $\varphi : M_2(K) \rightarrow M_1(K)$

$$a = a_{11} a_1 + a_{12} (a_1 e) + a_{21} (e a_1) + a_{22} (e a_1 e) \rightarrow \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix},$$

then clearly $\varphi$ is an algebra isomorphism over $K$. We also have

$$a \geq 0 \Rightarrow a_{11} a_1 \geq 0, a_{12} (a_1 e) \geq 0, a_{21} (e a_1) \geq 0, a_{22} (e a_1 e) \geq 0,$$

$$a_{11} a_1 \geq 0, a_{12} a_2 \geq 0, a_{21} a_2 \geq 0, a_{22} a_2 \geq 0 \text{ in } T_1,$$

$$a_{11} \geq 0, a_{12} \geq 0, a_{21} \geq 0, a_{22} \geq 0 \text{ in } K$$
Hence, the $\ell$-algebra $M_2(K)$ is isomorphic to the $\ell$-algebra $M_1(K)$ with the lattice order defined by $(a_j)_j \geq 0$ if each $a_j \geq 0$ in $K$. This completes the proof.

Finally, we provide an example to show that the assumption that $u$ is a $d$-element cannot be omitted in Theorem 3.1.

**Example 3.2** Let $K = \mathbb{Q}(\sqrt{2}) = \{(a + \beta \sqrt{2} \mid a, \beta \in \mathbb{Q}) \text{ and } u = 1 + \sqrt{2}\}$, then $(1, u)$ is a basis of vector space $K$ over $\mathbb{Q}$. Define an element $a + \beta u$ to be positive if $a \geq 0$ and $\beta \geq 0$. Since $u^2 = 1 + 2u$, $K$ becomes an $\ell$-field in which $u$ is not a $d$-element since $1 \land u = 0$ but $u \land u^2 = u \land (1 + 2u) = u \neq 0$. Another lattice order may be defined by the positive cone $P = \mathbb{Q}^+ 1 + \mathbb{Q}^+ \sqrt{2}$. Then $K^+ \subseteq P$. Now, we define a lattice order on $M_2(K)$ by calling a matrix $a = (a_j)$ positive if each $a_j \in P$. Since $K^+ \subseteq P$, $M_2(K)$ is an $\ell$-algebra over $K$. We notice that $e = e_{11} + e_{22} \geq 0$, and $I + e$ is not a $d$-element on $K$ since $1 \land u = 0$ but $(I + e) \land u(I + e) = I + e \neq 0$. However, the lattice order on $M_2(K)$ is not defined by the usual total order extension of $K^+$. 

Acknowledgement
The author thanks professor Yuehui Zhang (Shanghai Jiao Tong University) for valuable comments during the preparation of this paper.

Funding
The authors received no direct funding for this research.

Author details
Jingjing Ma
E-mail: ma@uhcl.edu
1 Department of Mathematics, University of Houston-Clear Lake, 2700 Bay Area Boulevard, Houston, TX 77059, USA.

Citation information
Cite this article as: Matrix $\ell$-algebras over $\ell$-fields, Jingjing Ma, Cogent Mathematics (2015), 2: 1053660.

References