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Fixed point theorems using control function in fuzzy metric spaces

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Abstract: The purpose of this paper is to obtain some results on existence of fixed points for contractive mappings in fuzzy metric spaces using control function. We prove our results on fuzzy metric spaces in the sense of George and Veeramani. Our results mainly generalize and extend the result of various authors, announced in the literature. As an application, a consequence theorem of integral type contraction is given in support of our result.

Subjects: Computer Science; Mathematics & Statistics; Science; Technology

Keywords: fuzzy metric spaces; fixed point; control function; contractive condition

AMS subject classifications: 54H25; 47H10; 47S40; 54A40

1. Introduction and preliminaries

The concept of fuzzy set was introduced by Zadeh (1965) and till then it has been developed extensively by many authors in different fields. The role of fuzzy topology in logic programming and algorithm has been recognized and applied on various programs to find more accurate results. In last 50 years, this theory has wide range of applications in diverse areas. The strong points about fuzzy mathematics are its fruitful applications, especially outside mathematics, such as in quantum particle physics studied by El Naschie (2004).

To use this concept in topology and analysis, Kramosil and Michalek (1975) have introduced the concept of fuzzy metric space using the concept of continuous triangular norm defined by Schweizer (1960). Most recently, Gregori, Morillas, and Sapena (2011) utilized the concept of fuzzy metric spaces to color image processing and also studied several interesting examples of fuzzy metrics in the sense of George and Veeramani (1994).

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Vishal Gupta , having more than 12 years of teaching experience, is working as an associate professor in the Department of Mathematics, Maharishi Markandeshwar University, India. He received his PhD degree in 2010. He has published one research book with an international publisher, and his immense contribution in journals of national and international repute is more than 50. He has presented more than 30 research papers in national and international conferences. He is also a reviewer of many prestigious professional bodies such as Mathematical Reviews, etc. His research interests are fixed point theory, fuzzy set theory and fuzzy mappings, topology, and differential and integral equations.

PUBLIC INTEREST STATEMENT

Now a days, fuzzy theory is one of the hot study in mathematical analysis. This theory has been generalized in different directions by several authors. One of the important generalizations of fuzzy theory is fuzzy metric spaces. Applications of this theory can be found in linguistics, decision-making, and clustering. In this paper, we shall prove some fixed point theorems under restricted distance functions. As an application, we have given consequence theorem of integral type contraction. Implementation and comparison of our result with other existing results are an open area under discussion for the future research.

Definition 1 (Schweizer, 1960) A binary operation $*:[0,1]\times[0,1]\rightarrow[0,1]$ is a **continuous triangular norm (t-norm)** if for all $a,b,c,e\in[0,1]$ the following conditions are satisfied:

(i) $*$ is commutative and associative,

(ii) $a*1=a$,

(iii) $*$ is continuous, and

(iv) $a*b\leq c*e$ whenever $a\leq c$ and $b\leq e$.

A fuzzy metric space in the sense of Kramosil and Michalek (1975) is defined as follows:

Definition 2 (Kramosil & Michalek, 1975) The triplet $(X, M, *)$ is said to be fuzzy metric space if X is an arbitrary set, $*$ is continuous t-norm, and M is fuzzy set on $X^2\times[0,\infty)$ satisfying the following conditions:

(i) $M(x,y,0)=0$,

(ii) $M(x,y,t)=1, \quad \forall\, t>0$ iff $x=y$,

(iii) $M(x,y,t)=M(y,x,t)$,

(iv) $M(x,y,t)*M(y,z,s)\leq M(x,z,t+s) \quad \forall\, x,y,z\in X$ and $t,s>0$,

(v) $M(x,y,.)\colon[0,\infty)\rightarrow[0,1]$ is left continuous, and

(vi) $\lim_{t\rightarrow\infty}M(x,y,t)=1 \quad \forall\, x,y\in X$.

The triplet $M(x,y,t)$ can be taken as the degree of nearness between x and y with respect to $t\geq 0$.

LEMMA 1 *For every $x,y\in X$, the mapping $M(x,y,.)$ is non-decreasing on $(0,\infty)$.*

Grabiec (1988) extended the fixed point theorem of Banach (1922) to fuzzy metric space in sense of Kramosil and Michalek (1975).

THEOREM 1 *(Grabiec, 1988) Let $(X, M, *)$ be a complete fuzzy metric space satisfying*

(i) $\lim_{t\rightarrow\infty}M(x,y,t)=1$, and

(ii) $M(Fx,Fy,kt)\geq M(x,y,t), \quad \forall\, x,\ y,\in X$,

where $0<k<1$. Then F has a unique fixed point.

Then Vasuki (1998) generalized Grabiecs result for common fixed point theorem for a sequence of mapping in a fuzzy metric space. Gregori and Sapena (2002) gave fixed point theorems for complete fuzzy metric space in the sense of George and Veeramani (1994) and also for Kramosil and Michaleks (1975) fuzzy metric space which are complete in Grabeics sense.

George and Veeramani (1994) modified the concept of fuzzy metric space introduced by Kramosil and Michalek (1975) with the help of t-norm and gave the following definition.

Definition 3 (George & Veeramani, 1994) The triplet $(X, M, *)$ is said to be fuzzy metric space if X is an arbitrary set, $*$ is continuous t-norm, and M is fuzzy set on $X^2\times[0,\infty)$ satisfying the following conditions:

(i) $M(x,y,t)>0$,

(ii) $M(x,y,t)=1, \quad \forall\, t>0$ iff $x=y$,

(iii) $M(x,y,t)=M(y,x,t)$,

(iv) $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s), \quad \forall\, x, y, z \in X$ and $t, s > 0$, and
(v) $M(x, y, .): [0, \infty) \rightarrow [0, 1]$ is continuous.

By introducing this definition, they also succeeded in introducing a Hausdorff topology on such fuzzy metric spaces which is widely used these days by researchers in their respective field of research. George and Veeramani (1994) have pointed out that the definition of Cauchy sequence given by Grabeic is weaker and hence it is essential to modify that definition to get better results in fuzzy metric space.

Consequently, some more metric fixed point results were generalized to fuzzy metric spaces by various authors such as Subrahmanyam (1995), Vasuki (1998), Saini, Gupta, and Singh (2007), Saini, Kumar, Gupta, and Singh (2008), Vijayaraju (2009), and Gupta and Mani (2014a, 2014b).

Now we give some important definitions and lemmas that are used in sequel.

Definition 4 (Grabiec, 1988) A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is said to be convergent to $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1 \quad \forall\, t > 0$.

Definition 5 (Grabiec, 1988) A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is called Cauchy sequence if $\lim_{n\to\infty} M(x_{n+p}, x_n, t) = 1 \quad \forall\, t > 0$ and each $p > 0$.

Definition 6 (Grabiec, 1988) A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence in X converges in X.

Example 1 (Gregori et al., 2011) Let (X, d) be a bounded metric space with $d(x, y) < k$ for all $x, y \in X$. Let $g: R^+ \rightarrow (k, \infty)$ be an increasing continuous function. Define a function M as

$$M(x, y, t) = 1 - \frac{d(x, y)}{g(t)},$$

then $(X, M, *)$ is a fuzzy metric space on X where $*$ is a Lukasievicz t-norm, i.e. $*(a, b) = \max\{a + b - 1, 0\}$.

Lemma 2 *If there exists $k \in (0, 1)$ such that $M(x, y, kt) \geq M(x, y, t)$ for all $x, y, \in X$ and $t \in (0, \infty)$, then $x = y$.*

In our result, we define a class Φ of all mappings $\xi: [0, 1] \rightarrow [0, 1]$ satisfying the following conditions:

(i) ξ is increasing on $[0, 1]$, and
(ii) $\xi(t) > t, \forall\, t \in (0, 1]$ and $\xi(t) = t$ if and only if $t = 1$.

In Section 2, we prove some fixed point theorems for contractive mappings in fuzzy metric spaces. We prove our results in fuzzy metric spaces in the sense of George and Veeramani (1994). Our result generalizes some relevant results in the literature.

2. Main results

Theorem 2 *Let $(X, M, *)$ be a complete fuzzy metric space and $f: X \rightarrow X$ be a mapping satisfying*

$$M(fx, fy, kt) \geq \xi\,\{\lambda(x, y, t)\}\,, \tag{1}$$

where

$$\lambda(x,y,t) = \min\left\{ M(x,y,t), M(x,fx,t), \frac{M(y,fy,t)[1+M(x,fx,t)]}{[1+M(x,y,t)]} \right\}, \tag{2}$$

for all $x, y, \in X$, $\xi \in \Phi$, and $k \in (0,1)$. Then f has a unique fixed point.

Proof Let $x \in X$ be any arbitrary point in X. Now construct a sequence $\{x_n\} \in X$ such that $fx_n = x_{n+1}$ for all $n \in N$.

Claim: $\{x_n\}$ *is a Cauchy sequence.*

Let us take $x = x_{n-1}$ and $y = x_n$ in Equation 1, we get

$$M(x_n, x_{n+1}, kt) = M(fx_{n-1}, fx_n, kt) \geq \xi\left\{\lambda(x_{n-1}, x_n, t)\right\}. \tag{3}$$

From Equation 2, we have

$$\begin{aligned}
&\lambda(x_{n-1}, x_n, t) \\
&\quad = \min\left\{ M(x_{n-1}, x_n, t), M(x_{n-1}, fx_{n-1}, t), \frac{M(x_n, fx_n, t)\,[1+M(x_{n-1}, fx_{n-1}, t)]}{[1+M(x_{n-1}, x_n, t)]} \right\} \\
&\quad = \min\left\{ M(x_{n-1}, x_n, t), M(x_{n-1}, x_n, t), \frac{M(x_n, x_{n+1}, t)\,[1+M(x_{n-1}, x_n, t)]}{[1+M(x_{n-1}, x_n, t)]} \right\} \\
&\quad = \min\left\{ M(x_n, x_{n+1}, t), M(x_{n-1}, x_n, t) \right\}.
\end{aligned}$$

Now if $M(x_n, x_{n+1}, t) \leq M(x_{n-1}, x_n, t)$, then from Equation 3

$$M(x_n, x_{n+1}, kt) \geq \xi\left\{M(x_n, x_{n+1}, t)\right\} > M(x_n, x_{n+1}, t).$$

Hence, our claim follows immediately from Lemma 2. Now suppose $M(x_n, x_{n+1}, t) \geq M(x_{n-1}, x_n, t)$, then again from Equation 3

$$M(x_n, x_{n+1}, kt) \geq \xi\left\{M(x_{n-1}, x_n, t)\right\} > M(x_{n-1}, x_n, t).$$

Now by simple induction, for all n and $t > 0$, we get

$$M(x_n, x_{n+1}, kt) \geq M\left(x, x_1, \frac{t}{k^{n-1}}\right). \tag{4}$$

Now for any positive integer s, we have

$$M(x_n, x_{n+s}, t) \geq M\left(x_n, x_{n+1}, \frac{t}{s}\right) * \cdots * M\left(x_{n+p-1}, x_{n+p}, \frac{t}{s}\right).$$

Using Equation 4, we get

$$M(x_n, x_{n+s}, t) \geq M\left(x, x_1, \frac{t}{sk^n}\right) * \cdots * M\left(x, x_1, \frac{t}{sk^n}\right).$$

Taking $\lim_{n\to\infty}$, we get

$$\lim_{n\to\infty} M\left(x_n, x_{n+s}, t\right) = 1. \tag{5}$$

This implies, $\{x_n\}$ is a Cauchy sequence; therefore, there exists a point $v \in X$ such that

$$\lim_{n\to\infty} x_n = v.$$

Claim: *v is a fixed point of f.*

Consider

$$\begin{aligned} M(v,fv,t) &\geq M(fx_n,fv,t) * M(v,x_{n+1},t) \\ &\geq \xi\left\{ \lambda\left(x_n,v,\frac{t}{2k}\right) \right\} * M(v,x_{n+1},t). \end{aligned} \tag{6}$$

Again from Equation 2

$$\begin{aligned} &\lambda\left(x_n,v,\frac{t}{2k}\right) \\ &\quad = \min\left\{ M\left(v,x_n,\frac{t}{2k}\right), M\left(x_n,fx_n,\frac{t}{2k}\right), \frac{M\left(v,fv,\frac{t}{2k}\right)\left[1+M\left(x_n,fx_n,\frac{t}{2k}\right)\right]}{\left[1+M\left(v,x_n,\frac{t}{2k}\right)\right]} \right\}. \end{aligned}$$

Taking $\lim_{n\to\infty}$ in above inequality, we get

$$\begin{aligned} &\lambda(v,v,\frac{t}{2k}) \\ &\quad = \min\left\{ M\left(v,v,\frac{t}{2k}\right), M\left(v,fv,\frac{t}{2k}\right), \frac{M\left(v,fv,\frac{t}{2k}\right)\left[1+M\left(v,fv,\frac{t}{2k}\right)\right]}{\left[1+M\left(v,v,\frac{t}{2k}\right)\right]} \right\} \\ &= \min\left\{ 1, M\left(v,fv,\frac{t}{2k}\right), M\left(v,fv,\frac{t}{2k}\right)\left[1+M\left(v,fv,\frac{t}{2k}\right)\right] \right\} \\ &= M\left(v,fv,\frac{t}{2k}\right). \end{aligned}$$

Hence from Equation 6, we get

$$M(v,fv,t) \geq \xi\left\{ M\left(v,fv,\frac{t}{2k}\right) \right\} * M\left(x_{n+1},v,t\right) > M\left(v,fv,\frac{t}{2k}\right) * M(x_{n+1},v,t). \tag{7}$$

On taking $\lim_{n\to\infty}$ in Equation 7 and using Lemma 2, we get $fv = v$.

Uniqueness: Now we show that v is a unique fixed point of f. Suppose not, then there exists a point $w \in X$ such that $fw = w$. Consider

$$1 \geq M(w,v,t) = M(fw,fv,t) \geq \xi\left\{ \lambda\left(w,v,\frac{t}{k}\right) \right\}, \tag{8}$$

where

$$\begin{aligned} &\lambda\left(w,v,\frac{t}{k}\right) \\ &\quad = \min\left\{ M\left(w,v,\frac{t}{k}\right), M\left(w,fw,\frac{t}{k}\right), \frac{M\left(v,fv,\frac{t}{k}\right)\left[1+M\left(w,fw,\frac{t}{k}\right)\right]}{\left[1+M\left(w,v,\frac{t}{k}\right)\right]} \right\} \\ &\quad = \min\left\{ M\left(w,v,\frac{t}{k}\right), M\left(w,w,\frac{t}{k}\right), \frac{2}{\left[1+M\left(w,v,\frac{t}{k}\right)\right]} \right\} \\ &\quad = \min\left\{ 1, M\left(w,v,\frac{t}{k}\right), \frac{2}{\left[1+M\left(w,v,\frac{t}{k}\right)\right]} \right\}. \end{aligned}$$

This implies either $\lambda\left(w,v,\frac{t}{k}\right) = 1$ or $\lambda\left(w,v,\frac{t}{k}\right) = M\left(w,v,\frac{t}{k}\right)$. Using it in Equation 8, we get $w = v$.

Thus, v is a unique fixed point of f. This completes the proof of Theorem 2. □

Corollary 1 *Let $(X, M, *)$ be a complete fuzzy metric space and $f: X \to X$ be a mapping satisfying*

$$M(fx, fy, kt) \geq \lambda(x, y, t),$$

where

$$\lambda(x,y,t) = \min\left\{ M(x,y,t), M(x,fx,t), \frac{M(y,fy,t)\left[1 + M(x,fx,t)\right]}{\left[1 + M(x,y,t)\right]} \right\},$$

for all $x, y, \in X$, and $k \in (0, 1)$. Then f has a unique fixed point.

The proof of the result follows immediately from Theorem 2 by taking $\xi(t) = t$.

3. Application

In this section, we give an application related to our result. Let us define $\Psi: [0, \infty) \to [0, \infty)$, as $\Psi(t) = \int_0^t \varphi(t)dt \quad \forall \quad t > 0$, be a non-decreasing and continuous function. Moreover, for each $\epsilon > 0$, $\varphi(\epsilon) > 0$. It also implies that $\varphi(t) = 0$ iff $t = 0$.

Theorem 3 *Let $(X, M, *)$ be a complete fuzzy metric space and $f: X \to X$ be a mapping satisfying*

$$\int_0^{M(fx,fy,kt)} \varphi(t)dt \geq \xi\left\{ \int_0^{\lambda(x,y,t)} \varphi(t)dt \right\},$$

where

$$\lambda(x,y,t) = \min\left\{ M(x,y,t), M(x,fx,t), \frac{M(y,fy,t)\left[1 + M(x,fx,t)\right]}{\left[1 + M(x,y,t)\right]} \right\},$$

for all $x, y, \in X$, $\varphi \in \Psi$, $\xi \in \Phi$, and $k \in (0, 1)$. Then f has a unique fixed point.

Proof By taking $\varphi(t) = 1$ and applying Theorem 2, we obtain the result. □

4. Conclusion

Our paper extends and generalizes the result of Grabeic (1988) and also some other results of literature such as Vasuki (1998), Gregori and Sapena (2002), and Gupta and Mani (2014a).

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