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APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

Local convergence for deformed Chebyshev-type method in Banach space under weak conditions

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Abstract: We present a local convergence analysis for deformed Chebyshev methods in order to approximate a solution of a nonlinear equation in a Banach space setting. Our methods include the Chebyshev and other high-order methods under hypotheses up to the first Fréchet derivative in contrast to earlier studies using hypotheses up to the second or third Fréchet derivative. The convergence ball and error estimates are given for these methods. Numerical examples are also provided in this study.

Subjects: Applied Mathematics; Computer Mathematics; Mathematics & Statistics; Non-Linear Systems; Science

Keywords: Chebyshev method; Banach space; convergence ball; local convergence

AMS subject classifications: 65D10; 65D99; 65G99; 47J25; 49M15

1. Introduction

Many problems in computational sciences and other disciplines can be brought in the form of

$$F(x) = 0 \tag{1.1}$$

where F is a Fréchet differentiable operator defined on a convex subset D of a Banach space X with values in a Banach space Y using mathematical modelling (Argyros, 1985, 2004, 2007; Argyros & Hilout, 2013; Gutiérrez & Hernández, 1997; Kantorovich & Akilov, 1982; Ortega & Rheinboldt, 1970).

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PUBLIC INTEREST STATEMENT

A large number of problems in applied mathematics, mathematical physics, mathematical economics, engineering and other areas are formulated as equations using mathematical modelling. The unknowns of these equations can be functions, vectors or real or complex numbers. In the present paper, we study the convergence of a fast sequence to the solution of an equation. The new results improve the results of earlier works. Solutions of these equations are very important, since they have a physical meaning.

In this study, we are concerned with approximating a solution x^* of Equation 1.1. The solutions of these equations in general can not be found in closed form, so one has to consider some iterative methods for solving Equation 1.1. The convergence analysis of iterative methods is usually based on two types: semi-local and local convergence analyses. The semi-local convergence analysis is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. In particular, the practice of numerical functional analysis for finding solution x^* of Equation 1.1 is essentially connected to variants of Newton's method. This method converges quadratically to x^* if the initial guess is close enough to the solution. Iterative methods of convergence order higher than two such as Chebyshev–Halley-type methods (Amat, Busquier, & Gutiérrez, 2003; Argyros, 2007; Argyros & Hilout, 2013; Candela & Marquina, 1990a, 1990b; Chun, Stanica, & Neta, 2011; Gutiérrez & Hernández, 1997, 1998; Hernández, 2001; Hernandez & Salanova, 2000; Kantorovich, 1982; Ortega & Rheinboldt, 1970; Parida & Gupta, 2008) require the evaluation of the second Fréchet derivative, which is very expensive in general. However, there are integral equations, where the second Fréchet derivative is diagonal by blocks and inexpensive (Gutiérrez & Hernández, 1997, 1998; Hernández, 2001; Hernández & Salanova, 2000) or for quadratic equations the second Fréchet derivative is constant (Argyros, 1985; Hernández & Salanova, 2000). Moreover, in some applications involving stiff systems (Argyros, 2004; Argyros & Hilout, 2013; Chun et al., 2011), high-order methods are useful. That is why we study the local convergence of Deformed Chebyshev Method (DCM) defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ z_n &= x_n + \alpha F'(x_n)^{-1}F(x_n) \\ H_n &= \frac{1}{\lambda} F'(x_n)^{-1} [F'(x_n + \lambda(z_n - x_n)) - F'(x_n)] \\ x_{n+1} &= x_n + \frac{1}{2} H_n (y_n - x_n) \end{aligned} \tag{1.2}$$

where x_0 is an initial point, $\lambda \in (0, 1]$ and $\alpha \in \mathbb{R}$ are given parameters. Deformed methods have been introduced to improve on the convergence order of Newton's method or Newton-like methods (Amat et al., 2003; Argyros, 1985, 2004, 2007; Argyros & Hilout, 2012, 2013; Candela & Marquina, 1990a, 1990b; Chun et al., 2011; Gutiérrez & Hernández, 1997, 1998; Hernández, 2001; Hernández & Salanova, 2000; Kantorovich, 1982; Ortega & Rheinboldt, 1970; Parida & Gupta, 2008; Wu & Zhao, 2007). In particular, DCM was proposed in Wu and Zhao (2007) as an alternative to the famous Chebyshev method (Amat et al., 2003; Argyros & Hilout, 2012, 2013; Candela & Marquina, 1990a, 1990b; Chun et al., 2011; Gutiérrez & Hernández, 1997, 1998; Hernández, 2001; Hernández & Salanova, 2000; Kantorovich, 1982; Ortega & Rheinboldt, 1970; Parida & Gupta, 2008; Wu & Zhao, 2007) defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n) \\ L_n &= F'(x_n)^{-1}F''(x_n)F'(x_n)^{-1}F(x_n) \\ x_{n+1} &= y_n + \frac{1}{2} L_n (y_n - x_n) \end{aligned} \tag{1.3}$$

Notice that the computation of the expensive in general second Fréchet derivative $F''(x_n)$ is required in method (1.3) but not in DCM.

The semilocal convergence analysis of DCM was given in Wu and Zhao (2007) under Lipschitz continuity conditions on up to the second Fréchet derivative in the special case when $\alpha = 1$ and $\lambda > 0$. In particular, the third order of convergence of DCM was shown in Wu and Zhao (2007) under these values of the parameters α and λ .

The usual conditions for the semi-local convergence of these methods are (C): There exist constants $\beta, \eta, \beta_1, \beta_2$ such that

$$(C_1) \quad \text{There exists } \Gamma_0 = F'(x_0)^{-1} \text{ and } \|\Gamma_0\| \leq \beta;$$

$$C_2 \quad \|\Gamma_0 F(x_0)\| \leq \eta;$$

$$C_3 \quad \|F''(x)\| \leq \beta_1 \text{ for each } x \in D;$$

$$C_4 \quad \|F''(x) - F''(y)\| \leq \beta_2 \|x - y\| \text{ for each } x, y \in D.$$

The local convergence conditions are similar but x_0 is x^* in (C_1) and (C_2) . There is a plethora of local and semi-local convergence results under the (C) conditions (Amat et al., 2003; Argyros, 1985, 2004, 2007; Argyros & Hilout, 2012, 2013; Candela & Marquina, 1990a, 1990b; Chun et al., 2011; Gutiérrez & Hernández, 1997, 1998; Hernández, 2001; Hernández & Salanova, 2000; Kantorovich, 1982; Ortega & Rheinboldt, 1970; Parida & Gupta, 2008; Wu & Zhao, 2007). The conditions (C_3) and (C_4) limit the applicability of these methods although only the first Fréchet derivative appears in these methods. Therefore, these usefull methods cannot be applied according to the earlier results. Therefore, the motivation for this study is to use these usefull methods (Wu & Zhao, 2007) in cases when (C_3) and (C_4) are not satisfied.

As a motivational example, let us define function f on $D = \left[-\frac{1}{2}, \frac{5}{2}\right]$ by

$$f(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Choose $x^* = 1$. We have that

$$\begin{aligned} f'(x) &= 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2 \\ f''(x) &= 6x \ln x^2 + 20x^3 - 12x^2 + 10x \\ f'''(x) &= 6 \ln x^2 + 60x^2 - 24x + 22 \end{aligned}$$

Notice that $f'''(x)$ does not satisfy (C_4) on D . Hence, the results depending on (C_4) cannot apply in this case. However, using Equations 2.7-2.10 that follow we have $f'(x^*) = 3$ and $f(x^*) = 0$, $p = 1$, $L_0 = L = 146.6629073$ and $M = 101.5578008$. Hence, the results of our Theorem 2.1 that follows can apply to solve equation $f(x) = 0$ using DCM. Hence, the applicability of DCM is expanded under our new conditions.

In the rest of this study, $U(w, q)$ and $\bar{U}(w, q)$ stand, respectively, for the open and closed ball in X with center $w \in X$ and of radius $q > 0$.

The paper is organized as follows: In Section 2, we present the local convergence of these methods. The numerical examples are given in the concluding Section 3.

2. Local convergence

In this section, we present the local convergence analysis of DCM. Let $L_0 > 0, L > 0, M > 0, \alpha \in \mathbb{R}, \lambda \in (0, 1]$ and $p \in [0, 1]$ be given parameters. It is convenient for the local convergence analysis that follows to introduce some functions and parameters.

Define functions on the interval $\left[0, \left(\frac{1}{L_0}\right)^p\right)$ by

$$g_1(t) = \frac{Lt^p}{(1+p)(1-L_0t^p)}$$

$$g_2(t) = g_1(t) + \frac{|1+\alpha|M}{1-L_0^p}$$

$$g_3(t) = \frac{|\alpha|^p L |\lambda|^{p-1} M^p t^p}{2(1-L_0t^p)^{1+p}}$$

$$g_4(t) = g_1(t) + \frac{|\alpha|^p L |\lambda|^{p-1} M^p t^p}{2(1-L_0t^p)^{1+p}} (1+g_1(t))$$

$$\bar{g}_4(t) = g_4(t) - 1$$

and parameters

$$r_1 = \left(\frac{1+p}{(1+p)L_0+L} \right)^{\frac{1}{p}} < \left(\frac{1}{L_0} \right)^{\frac{1}{p}}$$

and

$$r_2 = \left(\frac{(1+p)(1-M|1+\alpha|)}{(1+p)L_0+L} \right)^{\frac{1}{p}}$$

Suppose that

$$M|1+\alpha| < 1 \tag{2.1}$$

Then, r_2 is well defined and

$$0 < r_2 < r_1 < \left(\frac{1}{L_0} \right)^{\frac{1}{p}}$$

We also have that

$$0 \leq g_1(t) < 1$$

$$0 \leq g_2(t) < 1$$

and

$$g_3(t) \geq 0$$

Moreover, we have that $\bar{g}_4(0) = g_4(0) - 1 = 0 - 1 = -1 < 0$ and $\bar{g}_4(t) \rightarrow \infty$ as $t \rightarrow \left(\left(\frac{1}{L_0} \right)^{\frac{1}{p}} \right)^-$.

Then, it follows from the intermediate value theorem that function \bar{g}_4 has zeros in $\left(0, \left(\frac{1}{L_0} \right)^{\frac{1}{p}} \right)$.

Denote by r_4 the smallest such zero. Set

$$r = \min\{r_2, r_4\} \tag{2.2}$$

Then, we have that

$$0 \leq g_1(t) < 1 \tag{2.3}$$

$$0 \leq g_2(t) < 1 \tag{2.4}$$

$$g_3(t) \geq 0 \tag{2.5}$$

and

$$0 \leq g_4(t) < 1 \text{ for each } t \in [0, r] \tag{2.6}$$

Next, we present the local convergence result for DCM.

THEOREM 2.1 *Let $F: D \subseteq X \rightarrow Y$ be a Fréchet differentiable operator. Suppose that there exist $x^* \in D$, $L_0 > 0$, $L > 0$, $M > 0$, $\alpha \in \mathbb{R}$, $\lambda \in (0, 1]$ and $p \in (0, 1]$ such that for each $x, y \in D$*

$$M|1 + \alpha| < 1$$

$$F(x^*) = 0, \quad F'(x^*)^{-1} \in L(Y, X) \tag{2.7}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0 \|x - x^*\|^p \tag{2.8}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \leq L \|x - y\|^p \tag{2.9}$$

$$\|F'(x^*)^{-1}F'(x)\| \leq M \tag{2.10}$$

and

$$\bar{U}(x^*, r) \subseteq D \tag{2.11}$$

where r is given by Equation 2.2. Then, sequence $\{x_n\}$ generated by DCM for $x_0 \in U(x^*, r)$ is well defined, remains in $U(x^*, r)$ for each $n = 0, 1, 2, \dots$ and converges to x^* . Moreover, the following estimates hold for each $n = 0, 1, 2, \dots$

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r \tag{2.12}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r \tag{2.13}$$

$$\|\frac{1}{2}H_n\| \leq g_3(\|x_n - x^*\|) \tag{2.14}$$

and

$$\|x_{n+1} - x^*\| \leq g_4(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\| < r \tag{2.15}$$

where the “ g ” functions are defined above Theorem 2.1. Furthermore, if there exists $R \in [r, \frac{2}{L_0})$ such that $\bar{U}(x^*, R) \subseteq D$, then the limit point x^* is the only solution of equation $F(x) = 0$ in $U(x^*, r)$.

Proof We shall show estimates Equations 2.12–2.15 using mathematical induction. Firstly, we shall show that y_0, z_0, x_1 exist and lie inside $U(x^*, r)$. In order for us to achieve this, we must show that the inverses appearing in method DCM exist for $n = 0$. By hypothesis $x_0 \in U(x^*, r)$. Using the definition of radius r and Equation 2.8, we get that

$$\|F'(x^*)^{-1}(F'(x_0) - F'(x^*))\| \leq L_0 \|x - x^*\|^p < L_0 r^p < 1 \tag{2.16}$$

It follows from Equation 2.16 and the Banach Lemma on invertible operators (Argyros, 2007; Argyros & Hilout, 2013; Kantorovich, 1982; Ortega & Rheinboldt, 1970) that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x^*)^{-1}F'(x^*)\| \leq \frac{1}{1 - L_0 \|x - x^*\|^p} < \frac{1}{1 - L_0 r^p} \tag{2.17}$$

Moreover y_0, z_0 are well defined by first and second substep of DCM for $n = 0$. Using the first substep of DCM for $n = 0$, we also get that

$$\begin{aligned}
 y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\
 &= [F'(x_0)^{-1}F'(x^*)] \left[\int_0^1 F'(x^*)^{-1} \right. \\
 &\quad \left. \times [F'(x^* + \theta(x_0 - x^*)) - F'(x_0)](x_0 - x^*) d\theta \right]
 \end{aligned} \tag{2.18}$$

Then, by the definition of function g_1 , Equations 2.3, 2.9, 2.17 and 2.18, we obtain that

$$\begin{aligned}
 \|y_0 - x^*\| &\leq \|F'(x_0)^{-1}F'(x^*)\| \left\| \int_0^1 F'(x^*)^{-1} [F'(x^* + \theta(x_0 - x^*)) \right. \\
 &\quad \left. - F'(x_0)] d\theta \right\| \|x_0 - x^*\| \\
 &\leq \frac{L \|x_0 - x^*\|^{1+p}}{(1+p)(1 - L_0 \|x_0 - x^*\|)} \\
 &\leq g_1(\|x_0 - x^*\|) \|x_0 - x^*\| \\
 &< \|x_k - x^*\| < r
 \end{aligned}$$

which shows Equation 2.12 for $n = 0$ and $y_0 \in U(x^*, r)$. Similarly, using the second substep of DCM for $n = 0$, we get that

$$z_0 - x^* = x_0 - x^* - F'(x_0)^{-1}F(x_0) + (1 + \alpha)F'(x_0)^{-1}F(x_0) \tag{2.19}$$

Then, by Equations 2.4, 2.10, 2.17, 2.19 the definition of function g_2 and Equation 2.12 (for $n = 0$), we obtain, since $F(x_0) = \int_0^1 F'(x^* + \theta(x_0 - x^*)) (x_0 - x^*) d\theta$ that,

$$\begin{aligned}
 \|z_0 - x^*\| &\leq \|x_0 - x^* - F'(x_0)^{-1}F(x_0)\| \\
 &\quad + |1 + \alpha| \|F'(x_0)^{-1}F'(x^*)\| \\
 &\quad \times \left\| \int_0^1 F'(x^*)^{-1} F'(x^* + \theta(x_0 - x^*)) d\theta \right\| \|x_0 - x^*\| \\
 &\leq \left[g_1(\|x_0 - x^*\|) + \frac{|1 + \alpha|M}{1 - L_0 \|x_0 - x^*\|} \right] \|x_0 - x^*\| \\
 &= g_2(\|x_0 - x^*\|) \|x_0 - x^*\| < \|x_0 - x^*\| < r
 \end{aligned}$$

which shows Equation 2.13 for $n = 0$ and $z_0 \in U(x^*, r)$. We have by the definition of λ and Equations 2.12, 2.13 (for $n = 0$) that

$$\|x_0 - x^* + \lambda(z_0 - x_0)\| \leq |1 - \lambda| \|x_0 - x^*\| + |\lambda| \|z_0 - x^*\| < (|1 - \lambda| + |\lambda|)r \leq r$$

which shows that $x_0 + \lambda(z_0 - x_0) \in U(x^*, r)$ and H_0 is well defined. We need an estimate on $\|H_0\|$. Using the definition of H_0 , g_3 , Equations 2.17 and 2.9, we get in turn that

$$\begin{aligned}
 \left\| \frac{1}{2} H_0 \right\| &\leq \frac{1}{2|\lambda|} \|F'(x_0)^{-1}F'(x^*)\| \|F'(x^*)^{-1} [F'(x^* + \lambda(z_0 - x_0)) - F'(x_0)]\| \\
 &\leq \frac{1}{2|\lambda|} \frac{L|\lambda|^p \|z_0 - x_0\|^p}{1 - L_0 \|x_0 - x^*\|^p} \\
 &\leq \frac{|\alpha|^p L |\lambda|^{p-1} (\|F'(x_0)^{-1}F'(x^*)\| \|F'(x^*)^{-1}F(x_0)\|)^p}{2(1 - L_0 \|x_0 - x^*\|^p)} \\
 &\leq \frac{|\alpha|^p L |\lambda|^{p-1} M^p \|x_0 - x^*\|^p}{2(1 - L_0 \|x_0 - x^*\|^p)^{1+p}} \\
 &= g_3(\|x_0 - x^*\|)
 \end{aligned}$$

which shows Equation 2.14 for $n = 0$. Then, using the last substep of DCM for $n = 0$, we get

$$\begin{aligned} \|x_1 - x^*\| &\leq \|y_0 - x^*\| + \frac{1}{2}H_0\|y_0 - x_0\| \\ &\leq g_1(\|x_0 - x^*\|)\|x_0 - x^*\| + \frac{|\alpha|^p L |\lambda|^{p-1} M^p \|x_0 - x^*\|^p}{2(1 - L_0\|x_0 - x^*\|^p)^{1+p}} \\ &\quad \times (\|y_0 - x^*\| + \|x_0 - x^*\|) \\ &\leq [g_1(\|x_0 - x^*\|) + \frac{|\alpha|^p L |\lambda|^{p-1} \|x_0 - x^*\|^p}{2(1 - L_0\|x_0 - x^*\|^p)^{1+p}}] \\ &\quad \times (1 + g_1(\|x_0 - x^*\|))\|x_0 - x^*\| \\ &= g_4(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r \end{aligned}$$

which shows Equation 2.15 for $n = 0$ and $x_1 \in U(x^*, r)$. By simply replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1} in the preceding estimates we arrive at estimates Equations 2.12–2.15. Finally, using the estimate $\|x_{k+1} - x^*\| < \|x_k - x^*\| < r$, we deduce that $x_{k+1} \in U(x^*, r)$ and $\lim_{k \rightarrow \infty} x_k = x^*$.

To show the uniqueness part, let $B = \int_0^1 F'(y^* + \theta(x^* - y^*))d\theta$ for some $y^* \in \bar{U}(x^*, R)$ with $F(y^*) = 0$. Using Equation 2.8 with $p = 1$, we get that

$$\begin{aligned} |F'(x^*)^{-1}(B - F'(x^*))| &\leq \int_0^1 L_0|y^* + \theta(x^* - y^*) - x^*|d\theta \\ &\leq L_0 \int_0^1 (1 - \theta)|x^* - y^*|d\theta \leq \frac{L_0}{2}R < 1 \end{aligned} \tag{2.20}$$

It follows from Equation 2.30 and the Banach Lemma on invertible functions that B is invertible. Finally, from the identity $0 = F(x^*) - F(y^*) = B(x^* - y^*)$, we deduce that $x^* = y^*$. \square

Remark 2.2 (a) Condition Equation 2.8 can be dropped, since this condition follows from (A_3) . Notice, however that

$$L_0 \leq L \tag{2.21}$$

holds in general and $\frac{L}{L_0}$ can be arbitrarily large (Argyros, 1985, 2004, 2007; Argyros & Hilout, 2012, 2013) (see also the examples).

(b) In view of condition Equation 2.8 and the estimate

$$\begin{aligned} \|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}[F'(x) - F'(x^*)] + I\| \\ &\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \\ &\leq 1 + L_0\|x - x^*\|^p \end{aligned}$$

condition (2.10) can be dropped and M can be replaced by

$$M(t) = 1 + L_0 t^p \tag{2.22}$$

(c) The convergence ball of radius r_1 was given by us in Argyros (1985, 2004, 2007) and Argyros and Hilout (2013) for Newton's method under conditions Equations 2.8 and 2.9. Estimate $r_2 < r_1$ shows that the convergence ball of higher than two DCM methods are smaller than the convergence ball of radius r_1 of the quadratically convergent Newton's method. The convergence ball given by Ortega and Rheinboldt (1970) for Newton's method is

$$r_R = \frac{2}{3L} < r_1 \text{ (for } p = 1) \tag{2.23}$$

if $L_0 < L$ and $\frac{r_2}{r_1} \rightarrow \frac{1}{3}$ as $\frac{L_0}{L} \rightarrow 0$. Hence, we do not expect r to be larger than r_1 no matter how we choose L_0, L, M and α .

- (d) The local results can be used for projection methods such as Arnoldi's method, the generalized minimum residual method, the generalized conjugate method for combined Newton/finite projection methods and in connection to the mesh independence principle in order to develop the cheapest and most efficient mesh refinement strategy (Argyros, 1985, 2004, 2007; Argyros & Hilout, 2013; Kantorovich, 1982; Ortega & Rheinboldt, 1970).
- (e) The results can also be used to solve equations where the operator F' satisfies the autonomous differential equation (Argyros, 1985, 2004, 2007; Argyros & Hilout, 2013; Kantorovich, 1982; Ortega & Rheinboldt, 1970):

$$F'(x) = T(F(x))$$

where T is a known continuous operator. Since $F'(x^*) = T(F(x^*)) = T(0)$, $F''(x^*) = F'(x^*)T'(F(x^*)) = T(0)T'(0)$, we can apply the results without actually knowing the solution x^* . Let as an example $F(x) = e^x - 1$. Then, we can choose $T(x) = x + 1$ and $x^* = 0$.

- (f) It is worth noticing that DCM is not changing when we use the condition of Theorem 2.1 instead of the stronger (C) conditions used in the earlier mentioned works. We can also use the computational order of convergence (COC) (Argyros & Hilout, 2013; Hernández & Salanova, 2000) defined by

$$\xi = \ln \left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|} \right) / \ln \left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|} \right)$$

or the approximate COC

$$\xi_1 = \ln \left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|} \right) / \ln \left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|} \right)$$

since the bounds given in Theorem 2.1 may be very pessimistic. This way we also avoid the computation of the error bounds as given in the earlier studies where bounds on the Fréchet-derivatives higher than one are used.

- (g) The restriction $\lambda \in (0, 1]$ can be dropped, if Equation 2.11 is replaced by

$$U_1 = \bar{U}(x^*, (|\lambda| + |1 - \lambda|)r) \subseteq D \tag{2.24}$$

for $\lambda \in \mathbb{R} - \{0\}$. Indeed, we will then have

$$\begin{aligned} \|x_n + \lambda(y_n - x_n) - x^*\| &\leq |\lambda|\|x_n - x^*\| + |1 - \lambda|\|y_n - x^*\| \\ &\leq (|\lambda| + |1 - \lambda|)r \\ &\Rightarrow x_n + \lambda(y_n - x_n) \in U_1 \end{aligned}$$

3. Numerical examples

We present numerical examples where we compute the radii of the convergence balls.

Example 3.1 Let $X = Y = \mathbb{R}$. Define function F on $D = [1, 3]$ by

$$F(x) = \frac{2}{3}x^{\frac{3}{2}} - x \tag{3.1}$$

Then, $x^* = \frac{9}{4}$, $F'(x^*)^{-1} = 2$, $L_0 = 1 < L = 2$, $p = 0.5$, $\alpha = -0.6585$, $\lambda = 1$ and $M = 2(\sqrt{3} - 1)$, $r_1 = 0.7746$, $r_2 = 0.04629$, $r_4 = 0.0167$ and $r = 0.0167$.

Example 3.2 Let $X = Y = \mathbb{R}^3$, $D = \bar{U}(0, 1)$ and $B(x) = F''(x)$ for each $x \in D$. Define F on D for $v = x, y, z$ by

$$F(v) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z \right) \tag{3.2}$$

Then, the Fréchet derivative is given by:

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice that $x^* = (0, 0, 0)$, $F'(x^*) = F'(x^*)^{-1} = \text{diag}\{1, 1, 1\}$, $L_0 = e - 1 < L = M = e$, $p = 1$, $\alpha = -0.8161$, $\lambda = 0.5$. The values of $r_1 = 0.3245$, $r_2 = 0.1625$, $r_4 = 0.2291$ and $r = 0.1625$.

Example 3.3 Let $X = Y = C[0, 1]$, the space of continuous functions defined on $[0, 1]$ and be equipped with the max norm. Let $D = \bar{U}(0, 1)$ and $B(x) = F''(x)$ for each $x \in D$. Define function F on D by

$$F(\varphi)(x) = \varphi(x) - 5 \int_0^1 x\theta\varphi(\theta)^3 d\theta \tag{3.3}$$

we have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15 \int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \text{ for each } \xi \in D$$

Then, we get that $p = 1$, $x^* = 0$, $L_0 = 7.5$, $L = 15$, $\alpha = -0.9412$, $\lambda = 0.5$ and $M = M(t) = 1 + 7.5t$. The values of $r_1 = 0.0667$, $r_2 = 0.0333$, $r_4 = 0.0449$ and $r = 0.0333$.

Example 3.4 Returning back to the motivational example at the introduction of this study, we have $p = 1$, $L_0 = L = 146.6629073$, $M = 101.5578008$, $\alpha = -0.9951$, $\lambda = 0.5$. The values of $r_1 = 0.0045$, $r_2 = 0.0023$, $r_4 = 0.0034$ and $r = 0.0023$.

Example 3.5 Let us consider the nonlinear Hammerstein integral equation of the second kind (Argyros, 2007; Argyros & Hilout, 2013; Kantorovich, 1982; Ortega & Rheinboldt, 1970) given by

$$y(s) = w(s) + \int_a^b G(s, t)[y(t)^{1+p} + \mu y(t)]dt \text{ for each } s \in [a, b] \tag{3.4}$$

where w is a given continuous function such that $w(s) > 0$ for each $s \in [a, b]$, $\mu \in \mathbb{R}$, $p \in (0, 1]$ and the kernel G is a nonnegative continuous function on $[a, b] \times [a, b]$. Equation 3.4 appears in many studies in mathematical physics (radiative transfer, kinetic theory of gases, neuron transport) and other applied areas (Argyros, 1985, 2007; Argyros & Hilout, 2013; Gutiérrez & Hernández, 1997, 1998; Hernández & Salanova, 2000; Hernández, 2001; Kantorovich, 1982; Ortega & Rheinboldt, 1970; Wu & Zhao, 2007). The kernel G is defined by

$$G(s, t) = \begin{cases} \frac{(b-s)(t-a)}{b-a}, & t \leq s \\ \frac{b-a}{(s-a)(b-t)}, & s \leq t \end{cases} \tag{3.5}$$

It is well known that Equation 3.4 is equivalent to the following two-point boundary value problem;

$$\begin{aligned} y'' + y^{1+p} + \mu y &= 0 \\ y(a) = v(a), y(b) &= v(b) \end{aligned} \tag{3.6}$$

Next, we shall solve Equation 3.5 by first discretizing it and using DCM for $\mu = 0$, $p = \frac{1}{2}$, $a = 0$, $b = 1$ and $y(0) = y(1) = 0$. We divide the interval $[0, 1]$ into m subintervals and let $l = \frac{1}{m}$. Let us denote the points of subdivision by $u_0 = 0 < u_1 < \dots < u_m = 1$ with the corresponding values of the function

$y_0 = y(u_0), y_1 = y(u_1), \dots, y_m = y(u_m)$. A simple approximation for the second derivative at these points is given by

$$y_i'' \approx \frac{y_{i-1} - 2y_i + y_{i+1}}{l^2}, i = 1, 2, \dots, m - 1$$

Notice that $y_0 = 0$ and $y_m = 0$. Therefore, we obtain the following system of nonlinear equations

$$\begin{aligned} l^2 y_1^{\frac{3}{2}} - 2y_1 + y_2 &= 0 \\ y_{i-1} + l^2 y_i^{\frac{3}{2}} - 2y_i + y_{i+1} &= 0 \\ y_{m-2} + l^2 y_{m-1}^{\frac{3}{2}} - 2y_{m-1} &= 0, i = 2, 3, \dots, m - 1 \end{aligned} \tag{3.7}$$

Hence, we have an operator $F : \mathbb{R}^{m-1} \rightarrow \mathbb{R}^{m-1}$ whose Fréchet differential can be written as:

$$F'(y) = \begin{bmatrix} \frac{3}{2}l^2 y_1^{\frac{1}{2}} - 2 & 1 & 0 & \dots & \dots & 0 \\ 1 & \frac{3}{2}l^2 y_2^{\frac{1}{2}} - 2 & 1 & 0 & \dots & 0 \\ \vdots & & & & \vdots & \\ 0 & \dots & \dots & 1 & \frac{3}{2}l^2 y_{m-1}^{\frac{1}{2}} - 2 & 1 \end{bmatrix}$$

Notice that we may not be able to find the second Fréchet differential since it would involve terms of the form $y_i^{\frac{1}{2}}$ and they may not exist. Therefore, the famous Chebyshev method Equation 1.3 cannot be used. However, DCM method Equation 1.2 obtained from method Equation 1.3 can be used, since only the first Fréchet differential appears in this method. Moreover, the theory introduced in this paper applies. Indeed, let $x \in \mathbb{R}^{m-1}$. Define the norm to be $\|x\| = \max_{1 \leq j \leq m-1} \|x_j\|$. The corresponding norm on $A \in \mathbb{R}^{m-1} \times \mathbb{R}^{m-1}$ is $\|A\| = \max_{1 \leq j \leq m-1} \sum_{i=1}^{m-1} |a_{ji}|$. Then, for all $y, z \in \mathbb{R}^{m-1}$ for which $|y_i| > 0, |z_i| > 0, i = 1, 2, \dots, m - 1$:

$$\begin{aligned} \|F'(y) - F'(z)\| &= \left\| \text{diag} \left\{ \frac{3}{2} \left(y_i^{\frac{1}{2}} - z_i^{\frac{1}{2}} \right) \right\} \right\| \\ &= \frac{3}{2} l^2 \max_{1 \leq j \leq m-1} |y_j^{\frac{1}{2}} - z_j^{\frac{1}{2}}| \\ &\leq \frac{3}{2} l^2 \max_{1 \leq j \leq m-1} |y_j - z_j|^{\frac{1}{2}} \\ &= \frac{3}{2} l^2 \|y - z\|^{\frac{1}{2}} \end{aligned}$$

That is we can choose $L_0 = L = \frac{3}{2} l^2 \|F'(x^*)^{-1}\|$. We choose $m = 10$ leading to nine equations. Since a solution would vanish at the endpoints and be positive in the interior a reasonable choice of initial approximation seems to be $130 \sin \pi x$. Hence, we obtain the following initial vector for solving system Equation 3.7

$$y^0 = \begin{bmatrix} 4.01524e + 01 \\ 7.63785e + 01 \\ 1.05135e + 02 \\ 1.23611e + 02 \\ 1.29999e + 02 \\ 1.23675e + 02 \\ 1.05257e + 02 \\ 7.65462e + 01 \\ 4.03495e + 01 \end{bmatrix}$$

Then, after two iterations DCM for $\alpha = \lambda = 1$ gives us the solution x^* defined by

$$x^* = \begin{bmatrix} 3.35740e + 01 \\ 6.52027e + 01 \\ 9.15664e + 01 \\ 1.09168e + 02 \\ 1.15363e + 02 \\ 1.09168e + 02 \\ 9.15664e + 01 \\ 6.52027e + 01 \\ 3.35740e + 01 \end{bmatrix}$$

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