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Approximating positive solutions of nonlinear first order ordinary quadratic differential equations

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Abstract: In this paper, the authors prove the existence as well as approximations of the positive solutions for an initial value problem of first-order ordinary nonlinear quadratic differential equations. An algorithm for the solutions is developed and it is shown that the sequence of successive approximations converges monotonically to the positive solution of related quadratic differential equations under some suitable mixed hybrid conditions. We base our results on the Dhage iteration method embodied in a recent hybrid fixed-point theorem of Dhage (2014) in partially ordered normed linear spaces. An example is also provided to illustrate the abstract theory developed in the paper.

Subjects: Advanced Mathematics; Analysis - Mathematics; Differential Equations; Mathematics & Statistics; Operator Theory; Science

Keywords: quadratic differential equation; initial value problem; Dhage iteration method; approximate positive solution

AMS subject classifications: 34A12; 34A38

1. Introduction

Given a closed and bounded interval $J=[t_0,t_0+a]$, of the real line $\mathbb{R}$ for some $t_0,a\in\mathbb{R}$ with $t_0\geq 0, a>0$, consider the initial value problem (in short IVP) of first-order ordinary nonlinear quadratic differential equation, (in short HDE)

$$\left.\begin{array}{l}\frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right]+\lambda\left[\frac{x(t)}{f(t,x(t))}\right]=g(t,x(t)),\ t\in J,\\ x(t_0)=x_0\in\mathbb{R},\end{array}\right\}\tag{1.1}$$

ABOUT THE AUTHORS

The key research project of the authors of the paper is to prove existence and find the algorithms for different nonlinear equations that arise in mathematical analysis and allied areas of mathematics via newly developed Dhage iteration method. The quadratic differential equations form an important class in the theory of differential equations. In the present paper, it is shown that the new method is also applicable to such type of nonlinear quadratic differential equations for proving the existence as well as approximations of the solutions under mixed monotonic and geometric conditions.

PUBLIC INTEREST STATEMENT

It is known that many of the natural, physical, biological, and social processes or phenomena are governed by mathematical models of nonlinear differential equations. So if a person is engaged in the study of such complex universal phenomena and not having the knowledge of sophisticated nonlinear analysis of this paper, then one may convinced the use of the results of this paper, in particular when one comes across a certain dynamic process which is based on a mathematical model of quadratic differential equations. In such situations, the application of the results of the present paper yields numerical concrete solutions under some suitable natural conditions thereby which it is possible to improve the situation for better desired goals.

for $\lambda \in \mathbb{R}, \lambda > 0$, where $f : J \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$ and $g : J \times \mathbb{R} \to \mathbb{R}$ are continuous functions.

By a *solution* of the QDE (1.1), we mean a function $x \in C^1(J, \mathbb{R})$ that satisfies

(i) $t \mapsto \dfrac{x}{f(t,x)}$ is a continuously differentiable function for each $x \in \mathbb{R}$, and

(ii) x satisfies the equations in (1.1) on J,where $C(J, \mathbb{R})$ is the space of continuously differentiable real-valued functions defined on J.

The QDE (1.1) with $\lambda = 0$ is well known in the literature and is a hybrid differential equation with a quadratic perturbation of second type. Such differential equations can be tackled with the use of hybrid fixed-point theory (cf. Dhage 1999; 2013; 2014a). The special cases of QDE (1.1) have been discussed at length for existence as well as other aspects of the solutions under some strong Lipschitz and compactness-type conditions which do not yield any algorithm to determine the numerical solutions. See Dhage and Regan (2000), Dhage and Lakshmikantham (2010) and the references therein. Very recently, the study of approximation of the solutions for the hybrid differential equations is initiated in Dhage, Dhage, and Ntouyas (2014) via hybrid fixed-point theory. Therefore, it is of interest and new to discuss the approximations of solutions for the QDE (1.1) along the similar lines. This is the main motivation of the present paper and it is proved that the existence of the solutions may be proved via an algorithm based on successive approximations under weaker partial continuity and partial compactness-type conditions.

2. Auxiliary results

Unless otherwise mentioned, throughout this paper that follows, let E denotes a partially ordered real-normed linear space with an order relation $\preceq$ and the norm $\|\cdot\|$. It is known that E is *regular* if $\{x_n\}_{n\in\mathbb{N}}$ is a nondecreasing (resp. nonincreasing) sequence in E such that $x_n \to x^*$ as $n \to \infty$, then $x_n \preceq x^*$ (resp. $x_n \succeq x^*$) for all $n \in \mathbb{N}$. Clearly, the partially ordered Banach space, $C(J, \mathbb{R})$ is regular and the conditions guaranteeing the regularity of any partially ordered normed linear space E may be found in Nieto and Lopez (2005) and Heikkilä and Lakshmikantham (1994) and the references therein.

We need the following definitions in the sequel.

Definition 2.1 A mapping $\mathcal{T} : E \to E$ is called *isotone* or *nondecreasing* if it preserves the order relation $\preceq$, that is if $x \preceq y$ implies $\mathcal{T}x \preceq \mathcal{T}y$ for all $x, y \in E$.

Definition 2.2 (Dhage, 2010) A mapping $\mathcal{T} : E \to E$ is called *partially continuous* at a point $a \in E$ if for $\epsilon > 0$ there exists a $\delta > 0$ such that $\|\mathcal{T}x - \mathcal{T}a\| < \epsilon$ whenever x is comparable to a and $\|x - a\| < \delta$. $\mathcal{T}$ called partially continuous on E if it is partially continuous at every point of it. It is clear that if $\mathcal{T}$ is partially continuous on E, then it is continuous on every chain C contained in E.

Definition 2.3 A mapping $\mathcal{T} : E \to E$ is called *partially bounded* if $T(C)$ is bounded for every chain C in E. $\mathcal{T}$ is called *uniformly partially bounded* if all chains $\mathcal{T}(C)$ in E are bounded by a unique constant. $\mathcal{T}$ is called *bounded* if $T(E)$ is a bounded subset of E.

Definition 2.4 A mapping $\mathcal{T} : E \to E$ is called *partially compact* if $\mathcal{T}(C)$ is a relatively compact subset of E for all totally ordered sets or chains C in E. $\mathcal{T}$ is called *uniformly partially compact* if $\mathcal{T}(C)$ is a uniformly partially bounded and partially compact on E. $\mathcal{T}$ is called *partially totally bounded* if for any totally ordered and bounded subset C of E, $\mathcal{T}(C)$ is a relatively compact subset of E. If $\mathcal{T}$ is partially continuous and partially totally bounded, then it is called *partially completely continuous* on E.

Definition 2.5 (Dhage, 2009) The order relation $\preceq$ and the metric d on a nonempty set E are said to be *compatible* if $\{x_n\}_{n\in\mathbb{N}}$ is a monotone, that is, monotone nondecreasing or monotone nonincreasing sequence in E and if a subsequence $\{x_{n_k}\}_{n\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ converges to x^* implies that the whole sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x^*. Similarly, given a partially ordered normed linear space $(E, \preceq, \|\cdot\|)$, the order relation $\preceq$ and the norm $\|\cdot\|$ are said to be compatible if $\preceq$ and the metric d defined through the norm $\|\cdot\|$ are compatible.

Clearly, the set $\mathbb{R}$ of real numbers with usual order relation $\leq$ and the norm defined by the absolute value function $|\cdot|$ has this property. Similarly, the finite-dimensional Euclidean space $\mathbb{R}^n$ with usual componentwise order relation and the standard norm possesses the compatibility property.

Definition 2.6 (Dhage, 2010) An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-function, provided $\psi(0) = 0$. Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear space. A mapping $\mathcal{T} : E \to E$ is called *partially nonlinear $\mathcal{D}$-Lipschitz* if there exists a $\mathcal{D}$-function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi(\|x - y\|) \tag{2.1}$$

for all comparable elements $x, y \in E$. If $\psi(r) = k\,r, k > 0$, then $\mathcal{T}$ is called a partially Lipschitz with a Lipschitz constant k.

Let $(E, \preceq, \|\cdot\|)$ be a partially ordered normed linear algebra. Denote

$$E^+ = \left\{x \in E \mid x \succeq \theta, \text{ where } \theta \text{ is the zero element of } E\right\}$$

and

$$\mathcal{K} = \{E^+ \subset E \mid uv \in E^+ \text{ for all } u, v \in E^+\}. \tag{2.2}$$

The elements of the set $\mathcal{K}$ are called the positive vectors in E. The following lemma follows immediately from the definition of the set $\mathcal{K}$, which is oftentimes used in the hybrid fixed-point theory of Banach algebras and applications to nonlinear differential and integral equations.

LEMMA 2.1 *(Dhage, 1999) If $u_1, u_2, v_1, v_2 \in \mathcal{K}$ are such that $u_1 \preceq v_1$ and $u_2 \preceq v_2$, then $u_1u_2 \preceq v_1v_2$.*

Definition 2.7 An operator $\mathcal{T} : E \to E$ is said to be positive if the range $R(\mathcal{T})$ of $\mathcal{T}$ is such that $R(\mathcal{T}) \subseteq \mathcal{K}$.

The Dhage iteration principle or method (in short DIP or DIM) developed in Dhage (2010; 2013; 2014a) may be rephrased as "monotonic convergence of the sequence of successive approximations to the solutions of a nonlinear equation beginning with a lower or an upper solution of the equation as its initial or first approximation" and which forms a useful tool in the subject of existence theory of nonlinear analysis. The Dhage iteration method is different from other iterations methods and embodied in the following applicable hybrid fixed-point theorem of Dhage (2014b), which is the key tool for our work contained in the present paper. A few other hybrid fixed-point theorems containing the Dhage iteration principle appear in Dhage (2010; 2013; 2014a; 2014b).

THEOREM 2.1 *Let $(E, \preceq, \|\cdot\|)$ be a regular partially ordered complete normed linear algebra such that the order relation $\preceq$ and the norm $\|\cdot\|$ in E are compatible in every compact chain of E. Let $\mathcal{A}, \mathcal{B} : E \to \mathcal{K}$ be two nondecreasing operators such that*

(a) $\mathcal{A}$ is partially bounded and partially nonlinear $\mathcal{D}$-Lipschitz with $\mathcal{D}$-function $\psi_{\mathcal{A}}$,
(b) $\mathcal{B}$ is partially continuous and uniformly partially compact,
(c) $M\psi_{\mathcal{A}}(r) < r, r > 0$, where $M = \sup\{\|\mathcal{B}(C)\| : C \in \mathcal{P}_{ch}(E)\}$, and
(d) there exists an element $x_0 \in X$ such that $x_0 \preceq \mathcal{A}x_0\,\mathcal{B}x_0$ or $x_0 \succeq \mathcal{A}x_0\,\mathcal{B}x_0$.

Then the operator equation

$$\mathcal{A}x\,\mathcal{B}x = x \tag{2.3}$$

has a positive solution x^* in E and the sequence $\{x_n\}$ of successive iterations defined by $x_{n+1} = \mathcal{A}x_n\,\mathcal{B}x_n$, $n = 0, 1, \ldots$; converges monotonically to x^*.

Remark 2.1 The compatibility of the order relation $\preceq$ and the norm $\|\cdot\|$ in every compact chain of E is held if every partially compact subset S of E possesses the compatibility property with respect to $\preceq$ and $\|\cdot\|$. This simple fact is used to prove the desired characterization of the positive solution of the QDE (1.1) defined on J.

3. Main results

The QDE (1.1) is considered in the function space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J. We define a norm $\|\cdot\|$ and the order relation $\leq$ in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t\in J} |x(t)| \tag{3.1}$$

and

$$x \leq y \iff x(t) \leq y(t) \tag{3.2}$$

for all $t \in J$, respectively. Clearly, $C(J, \mathbb{R})$ is a Banach algebra with respect to above supremum norm and is also partially ordered w.r.t. the above partially order relation $\leq$. It is known that the partially ordered Banach algebra $C(J, \mathbb{R})$ has some nice properties w.r.t. the above order relation in it. The following lemma follows by an application of Arzelá–Ascoli theorem.

LEMMA 3.1 *Let $\big(C(J, \mathbb{R}), \leq, \|\cdot\|\big)$ be a partially ordered Banach space with the norm $\|\cdot\|$ and the order relation $\leq$ defined by (3.1) and (3.2), respectively. Then, $\|\cdot\|$ and $\leq$ are compatible in every partially compact subset of $C(J, \mathbb{R})$.*

Proof The proof of the lemma is given in Dhage and Dhage (in press). Since it is not well known, we give the details of proof for the sake of completeness. Let S be a partially compact subset of $C(J, \mathbb{R})$ and let $\{x_n\}$ be a monotone nondecreasing sequence of points in S. Then, we have

$$x_1(t) \leq x_2(t) \leq \cdots \leq x_n(t) \leq \cdots \tag{ND}$$

for each $t \in J$.

Suppose that a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ is convergent and converges to a point x in S. Then the subsequence $\{x_{n_k}(t)\}$ of the monotone real sequence $\{x_n(t)\}$ is convergent. By monotone characterization, the whole sequence $\{x_n(t)\}$ is convergent and converges to a point $x(t)$ in $\mathbb{R}$ for each $t \in J$. This shows that the sequence $\{x_n(t)\}$ converges pointwise in S. To show the convergence is uniform, it is enough to show that the sequence $\{x_n(t)\}$ is equicontinuous. Since S is partially compact, every chain or totally ordered set and consequently $\{x_n\}$ is an equicontinuous sequence by Arzelá–Ascoli theorem. Hence $\{x_n\}$ is convergent and converges uniformly to x. As a result, $\|\cdot\|$ and $\leq$ are compatible in S. This completes the proof.

We need the following definition in what follows.

Definition 3.1 A function $u \in C^1(J, \mathbb{R})$ is said to be a lower solution of the QDE (1.1) if the function $t \mapsto \dfrac{u(t)}{f(t, u(t))}$ is continuously differentiable and satisfies

$$\left.\begin{aligned} \frac{d}{dt}\left[\frac{u(t)}{f(t,u(t))}\right] &+ \lambda\left[\frac{u(t)}{f(t,u(t))}\right] \le g(t,u(t)), \\ &u(t_0) \le x_0 \end{aligned}\right\}$$

for all $t \in J$. Similarly, a function $v \in C^1(J,\mathbb{R})$ is said to be an upper solution of the QDE (1.1) if it satisfies the above property and inequalities with reverse sign.

We consider the following set of assumptions in what follows:

(A_0) The map $x \mapsto \dfrac{x}{f(t,x)}$ is injection for each $t \in J$.
(A_1) f defines a function $f{:}J \times \mathbb{R} \to \mathbb{R}_+$.
(A_2) There exists a constant $M_f > 0$ such that $0 < f(t,x) \le M_f$ for all $t \in J$ and $x \in \mathbb{R}$.
(A_3) There exists a $\mathcal{D}$-function ϕ, such that

$$0 \le f(t,x) - f(t,y) \le \phi(x-y),$$

for all $t \in J$ and $x, y \in \mathbb{R}$, $x \ge y$.
(B_1) g defines a function $g{:}J \times \mathbb{R} \to \mathbb{R}_+$.
(B_2) There exists a constant $M_g > 0$ such that $g(t,x) \le M_g$ for all $t \in J$ and $x \in \mathbb{R}$.
(B_3) $g(t,x)$ is nondecreasing in x for all $t \in J$.
(B_4) The QDE (1.1) has a lower solution $u \in C^1(J,\mathbb{R})$.

Remark 3.1 Notice that Hypothesis (A_0) holds in particular if the function $x \mapsto \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$ for each $t \in J$.

LEMMA 3.2 *Suppose that hypothesis (A_0) holds. Then a function $x \in C(J,\mathbb{R})$ is a solution of the QDE (1.1), if and only if it is a solution of the nonlinear quadratic integral equation (in short QIE),*

$$x(t) = \left[f(t,x(t))\right]\left(\frac{ce^{-\lambda t}}{f(t_0,x_0)} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s,x(s))\,ds\right) \tag{3.3}$$

for all $t \in J$, where $c = x_0 e^{\lambda t_0}$.

THEOREM 3.1 *Assume that hypotheses (A_0)–(A_3) and (B_1)–(B_4) hold. Furthermore, assume that*

$$\left(\left|\frac{x_0}{f(t_0,x_0)}\right| + M_g\, a\right)\phi(r) < r,\ \ r < 0, \tag{3.4}$$

then the QDE (1.1) has a positive solution x^* defined on J and the sequence $\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by

$$x_{n+1}(t) = \left[f(t,x_n(t))\right]\left(\frac{ce^{-\lambda t}}{f(t_0,x_0)} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s,x_n(s))\,ds\right) \tag{3.5}$$

for $t \in \mathbb{R}$, where $x_1 = u$, converges monotonically to x^*.

Proof Set $E = C(J,\mathbb{R})$ Then, by Lemma 3.1, every compact chain in E possesses the compatibility property with respect to the norm $\|\cdot\|$ and the order relation $\le$ in E.

Define two operators $\mathcal{A}$ and $\mathcal{B}$ on E by

$$\mathcal{A}x(t) = f(t, x(t)), \quad t \in J, \tag{3.6}$$

and

$$\mathcal{B}x(t) = \frac{ce^{-\lambda t}}{f(t_0, x_0)} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x(s))\, ds, \quad t \in J. \tag{3.7}$$

From the continuity of the integral, it follows that $\mathcal{A}$ and $\mathcal{B}$ define the maps $\mathcal{A}, \mathcal{B} : E \to E$. Now by Lemma 3.2, the QDE (1.1) is equivalent to the operator equation

$$\mathcal{A}x(t)\mathcal{B}x(t) = x(t), \quad t \in J. \tag{3.8}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.1. This is achieved in the series of following steps.

Step I: $\mathcal{A}$and $\mathcal{B}$ are nondecreasing on E.

Let $x, y \in E$ be such that $x \geq y$. Then by hypothesis (A_3), we obtain

$$\mathcal{A}x(t) = f(t, x(t)) \geq f(t, y(t)) = \mathcal{A}y(t)$$

for all $t \in J$. This shows that $\mathcal{A}$ is nondecreasing operator on E into E. Similarly using hypothesis (B_3), it is shown that the operator $\mathcal{B}$ is also nondecreasing on E into itself. Thus, $\mathcal{A}$ and $\mathcal{B}$ are nondecreasing positive operators on E into itself.

Step II: $\mathcal{A}$ is partially bounded and partially $\mathcal{D}$-Lipschitz on E.

Let $x \in E$ be arbitrary. Then by (A_2),

$$|\mathcal{A}x(t)| \leq |f(t, x(t))| \leq M_f$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{A}x\| \leq M_f$ and so, $\mathcal{A}$ is bounded. This further implies that $\mathcal{A}$ is partially bounded on E.

Next, let $x, y \in E$ be such that $x \geq y$. Then,

$$|\mathcal{A}x(t) - \mathcal{A}y(t)| = |f(t, x(t)) - f(t, y(t))| \leq \phi(|x(t) - y(t)|) \leq \phi(\|x - y\|)$$

for all $t \in J$. Taking supremum over t, we obtain $\|\mathcal{A}x - \mathcal{A}y\| \leq \phi(\|x - y\|)$ for all $x, y \in E$, $x \geq y$. Hence, $\mathcal{A}$ is a partial nonlinear $\mathcal{D}$-LIpschitz on E which further implies that $\mathcal{A}$ is a partially continuous on E.

Step III: *$\mathcal{B}$ is partially continuous on E.*

Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in a chain C of E such that $x_n \to x$ for all $n \in \mathbb{N}$. Then, by dominated convergence theorem, we have

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}x_n(t) &= \lim_{n\to\infty} \frac{ce^{-\lambda t}}{f(t_0, x_0)} + \lim_{n\to\infty} \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x_n(s))\, ds \\
&= \frac{ce^{-\lambda t}}{f(t_0, x_0)} + \int_{t_0}^{t} e^{-\lambda(t-s)} \left[\lim_{n\to\infty} g(s, x_n(s))\right] ds \\
&= \frac{ce^{-\lambda t}}{f(t_0, x_0)} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x(s))\, ds \\
&= \mathcal{B}x(t)
\end{aligned}$$

for all $t \in J$. This shows that $\mathcal{B}x_n$ converges monotonically to $\mathcal{B}x$ pointwise on J.

Next, we will show that $\{\mathcal{B}x_n\}_{n\in\mathbb{N}}$ is an equicontinuous sequence of functions in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then

$$\begin{aligned}
|\mathcal{B}x_n(t_2) - \mathcal{B}x_n(t_1)| &\le \left| \frac{ce^{-\lambda t_1}}{f(t_0, x_0)} - \frac{ce^{-\lambda t_2}}{f(t_0, x_0)} \right| \\
&\quad + \left| \int_{t_0}^{t_1} e^{-\lambda(t_1 - s)} g(s, x_n(s))\, ds - \int_{t_0}^{t_1} e^{-\lambda(t_2 - s)} g(s, x_n(s))\, ds \right| \\
&\quad + \left| \int_{t_0}^{t_1} e^{-\lambda(t_2 - s)} g(s, x_n(s))\, ds - \int_{t_0}^{t_2} e^{-\lambda(t_2 - s)} g(s, x_n(s))\, ds \right| \\
&\le \left| \frac{ce^{-\lambda t_1}}{f(t_0, x_0)} - \frac{ce^{-\lambda t_2}}{f(t_0, x_0)} \right| + \left| \int_{t_0}^{t_1} \left| e^{-\lambda(t_1 - s)} - e^{-\lambda(t_2 - s)} \right| |g(s, x_n(s))|\, ds \right| \\
&\quad + \left| \int_{t_2}^{t_1} |g(s, x_n(s))\, ds| \right| \\
&\le \left| \frac{ce^{-\lambda t_1}}{f(t_0, x_0)} - \frac{ce^{-\lambda t_2}}{f(t_0, x_0)} \right| + M_g \int_{t_0}^{t_0 + a} \left| e^{-\lambda(t_1 - s)} - e^{-\lambda(t_2 - s)} \right| ds \\
&\quad + M_g |t_1 - t_2| \\
&\to 0 \quad as \quad t_2 - t_1 \to 0
\end{aligned}$$

uniformly for all $n \in \mathbb{N}$. This shows that the convergence $\mathcal{B}x_n \to \mathcal{B}x$ is uniform and hence $\mathcal{B}$ is partially continuous on E.

Step IV: $\mathcal{B}$ is uniformly partially compact operator on E.

Let C be an arbitrary chain in E. We show that $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set in E. First, we show that $\mathcal{B}(C)$ is uniformly bounded. Let $y \in \mathcal{B}(C)$ be any element. Then there is an element $x \in C$, such that $y = \mathcal{B}x$. Now, by hypothesis (B_2),

$$\begin{aligned}
|y(t)| &\le \left| \frac{ce^{-\lambda t}}{f(t_0, x_0)} + \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x(s))\, ds \right| \\
&\le \left| \frac{ce^{-\lambda t}}{f(t_0, x_0)} \right| + \left| \int_{t_0}^{t} e^{-\lambda(t-s)} g(s, x(s))\, ds \right| \\
&\le \left| \frac{x_0}{f(t_0, x_0)} \right| + \int_{t_0}^{t_0 + a} |g(s, x(s))|\, ds \\
&\le \left| \frac{x_0}{f(t_0, x_0)} \right| + M_g\, a = M
\end{aligned}$$

for all $t \in J$. Taking supremum over t, we obtain $\|y\| = \|\mathcal{B}x\| \le M$ for all $y \in \mathcal{B}(C)$. Hence, $\mathcal{B}(C)$ is a uniformly bounded subset of E. Moreover, $\|\mathcal{B}(C)\| \le M$ for all chains C in E. Hence, $\mathcal{B}$ is a uniformly partially bounded operator on E.

Next, we will show that $\mathcal{B}(C)$ is an equicontinuous set in E. Let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for any $y \in \mathcal{B}(C)$, one has

$$\begin{aligned}
|y(t_2) - y(t_1)| &= |\mathcal{B}x(t_2) - \mathcal{B}x(t_1)| \\
&\le \left| \frac{ce^{-\lambda t_1}}{f(t_0, x_0)} - \frac{ce^{-\lambda t_2}}{f(t_0, x_0)} \right| \\
&\quad + \left| \int_{t_0}^{t_1} e^{-\lambda(t_1 - s)} g(s, x(s))\, ds - \int_{t_0}^{t_1} e^{-\lambda(t_2 - s)} g(s, x(s))\, ds \right| \\
&\quad + \left| \int_{t_0}^{t_1} e^{-\lambda(t_2 - s)} g(s, x(s))\, ds - \int_{t_0}^{t_2} e^{-\lambda(t_2 - s)} g(s, x(s))\, ds \right| \\
&\le \left| \frac{ce^{-\lambda t_1}}{f(t_0, x_0)} - \frac{ce^{-\lambda t_2}}{f(t_0, x_0)} \right| + \left| \int_{t_0}^{t_1} \left| e^{-\lambda(t_1 - s)} - e^{-\lambda(t_2 - s)} \right| |g(s, x(s))|\, ds \right| \\
&\quad + \left| \int_{t_2}^{t_1} |g(s, x(s))\, ds| \right| \\
&\le \left| \frac{ce^{-\lambda t_1}}{f(t_0, x_0)} - \frac{ce^{-\lambda t_2}}{f(t_0, x_0)} \right| + M_g \int_{t_0}^{t_0 + a} \left| e^{-\lambda(t_1 - s)} - e^{-\lambda(t_2 - s)} \right| ds \\
&\quad + M_g \, |t_1 - t_2| \\
&\to 0 \quad as \quad t_2 - t_1 \to 0
\end{aligned}$$

uniformly for all $y \in \mathcal{B}(C)$. Hence $\mathcal{B}(C)$ is an equicontinuous subset of E. Now, $\mathcal{B}(C)$ is a uniformly bounded and equicontinuous set of functions in E, so it is compact. Consequently, $\mathcal{B}$ is a uniformly partially compact operator on E into itself.

Step V: u satisfies the operator inequality $u \le \mathcal{A}u\, \mathcal{B}u$.

By hypothesis (B_4), the QDE (1.1) has a lower solution u defined on J. Then, we have

$$\left.\begin{aligned}
\tfrac{d}{dt}\left[\tfrac{u(t)}{f(t,u(t))}\right] \quad &+ \lambda\left[\tfrac{u(t)}{f(t,u(t))}\right] \le g(t,\, u(t)), \\
&u(t_0) \le x_0
\end{aligned}\right\} \tag{3.9}$$

for all $t \in J$. Multiplying the above inequality (3.9) by the integrating factor $e^{\lambda t}$, we obtain

$$\left(e^{\lambda t} \frac{u(t)}{f(t, u(t))} \right)' \le e^{\lambda t} g(t, u(t)) \tag{3.10}$$

for all $t \in J$. A direct integration of (3.10) from t_0 to t yields

$$u(t) \le [f(t, u(t))] \left(\frac{ce^{-\lambda t}}{f(t_0, x_0)} + \int_{t_0}^{t} e^{-\lambda(t - s)} g(s, u(s))\, ds \right) \tag{3.11}$$

for all $t \in J$. From definitions of the operators $\mathcal{A}$ and $\mathcal{B}$, it follows that $u(t) \le \mathcal{A}u(t)\, \mathcal{B}u(t)$, for all $t \in J$. Hence $u \le \mathcal{A}u\, \mathcal{B}u$.

Step VI: $\mathcal{D}$-function ϕ satisfies the growth condition $M\psi_{\mathcal{A}}(r) < r, r > 0$.

Finally, the $\mathcal{D}$-function ϕ of the operator $\mathcal{A}$ satisfies the inequality given in hypothesis (d) of Theorem 2.1. Now from the estimate given in Step IV, it follows that

$$M\psi_{\mathcal{A}}(r) \le \left(\left| \frac{x_0}{f(t_0, x_0)} \right| + M_g\, a \right) \phi(r) < r$$

for all $r > 0$.

Thus, $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 2.1 and we apply it to conclude that the operator equation $\mathcal{A}x\,\mathcal{B}x = x$ has a solution. Consequently the integral Equation 3.3 and the QDE (1.1) has a solution x^* defined on J. Furthermore, the sequence $\{x_n\}_{n=1}^{\infty}$ of successive approximations defined by (3.5) converges monotonically to x^*. This completes the proof.

Remark 3.2 The conclusion of Theorem 3.1 also remains true if we replace the hypothesis ($\mathbf{B}_4$) with the following:

(B_4') The QDE (1.1) has an upper solution $v \in C^1(J, \mathbb{R})$.

The proof under this new hypothesis is similar to the proof of Theorem 3.1 with appropriate modifications.

Example 3.1 Given a closed and bounded interval $J = [0, 1]$, consider the IVP of QDE,

$$\left.\begin{array}{l} \frac{d}{dt}\left[\frac{x(t)}{f(t,x(t))}\right] = \frac{1}{4}\left[2 + \tanh x(t)\right], \ t \in J, \\ x(0) = 0 \in \mathbb{R} \end{array}\right\} \tag{3.12}$$

where the functions $f, g : J \times \mathbb{R} \to \mathbb{R}$ are defined as

$$f(t,x) = \begin{cases} 1, & \text{if } \ x \le 0, \\ 1 + x, & \text{if } \ 0 < x < 3, \\ 4, & \text{if } \ , x \ge 3 \end{cases}$$

and

$$g(t,x) = \frac{1}{4}\left[2 + \tanh x\right]$$

Clearly, the functions f and g are continuous on $J \times \mathbb{R}$ into $\mathbb{R}_+$. The function f satisfies the hypothesis (A_3) with $\phi(r) = r$. To see this, we have

$$0 \le f(t,x) - f(t,y) \le x - y$$

for all $x, y \in \mathbb{R}$, $x \ge y$. Therefore, $\phi(r) = r$. Moreover, the function $f(t,x)$ is positive and bounded on $J \times \mathbb{R}$ with bound $M_f = 4$ and so the hypothesis (A_2) is satisfied. Again, since g is positive and bounded on $J \times \mathbb{R}$ by $M_g = \dfrac{3}{4}$, the hypothesis (B_2) holds. Furthermore, $g(t,x)$ is nondecreasing in x for all $t \in J$, and thus hypothesis (B_3) is satisfied. Also, condition (3.4) of Theorem 3.1 is held. Finally, the QDE (3.12) has a lower solution $u(t) = \dfrac{t}{4}$ defined on J, thus all hypotheses of Theorem 3.1 are satisfied. Hence, we apply Theorem 3.1 and conclude that the QDE (3.12) has a solution x^* defined on J and the sequence $\{x_n\}_{n=1}^{\infty}$ defined by

$$x_{n+1}(t) = \frac{1}{4}\left[f(t, x_n(t))\right]\left(\int_0^t \left[2 + \tan h x_n(s)\right]\, ds\right) \tag{3.13}$$

for all $t \in J$, where $x_1 = u$, converges monotonically to x^*.

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