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STATISTICS | RESEARCH ARTICLE

On the Bayesianity of minimum risk equivariant estimator for location or scale parameters under a general convex and invariant loss function

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Abstract: The Minimum Risk Equivariant (MRE), estimator is a widely used estimator which has several well-known theoretical and practical properties. It is well known that for the square error and absolute error loss functions, the MRE estimator is a generalized Bayes estimator. This article investigates the potential Bayesianity (or generalized Bayesianity) of the MRE estimator under a general convex and invariant loss function, $\rho(\cdot)$, for estimating the location and scale parameters of an unimodal density function.

Subjects: Mathematical Statistics; Mathematics & Statistics; Statistics; Science; Statistics & Probability

Keywords: Bayes estimator; the Fourier transform; the Minimum Risk Equivariant (MRE) estimator; convex and invariant loss functions

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1. Introduction

Compare with the uniform minimum variance unbiased estimator, the Minimum Risk Equivariant (MRE) estimator: (1) typically exists not only for convex loss function but also even for non-restricted loss functions and (2) does not need to consider randomized estimators (Lehmann & Casella, 1998, p. 156). Moreover, the MRE estimator is a widely used estimator which has several well-known theoretical (such as minimaxity and admissibility under some certain conditions) and practical properties. The MRE estimator has a wide range of applications in Finite sampling framework (Chandrasekar & Sajesh, 2013; Ledoit & Wolf, 2013), Reliability (Chandrasekar & Sajesh, 2013; Wei, Song, Yan, & Mao, 2000), Regression and non-linear models (Grafarend, 2006; Hallin & Jurečková, 2012), Contingency tables (Lehmann & Casella, 1998), Economic Forecasting (Elliott & Timmermann, 2013), etc.

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PUBLIC INTEREST STATEMENT

This article investigates the potential Bayesianity (or generalized Bayesianity) of the MRE estimator under a general convex and invariant loss function, $\rho(\cdot)$, for estimating the location and scale parameters of an unimodal density function.

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It is well known that for the square error and the absolute error loss functions, the MRE estimator is a generalized Bayes estimator. This article extends this fact to a class of convex and invariant loss functions for unimodal location (or scale) density functions. This fact has been proven remarkably useful in solving a variety of problems in statistics. Subjects for which the problem is applicable range from truncated data (Manrique-Vallier & Reiter, 2014), imputation omitted data due to undercounting (Rubin, Gelman, & Meng, 2004, §12), Capture–recapture Estimation (Mitchell, 2014), etc.

The problem of finding a prior distribution that its corresponding Bayes estimator under a given loss function coincides with a given estimator started by Lehmann (1951). In his seminal paper, he considered a situation that a Bayes estimator under the squared error loss function is an unbiased estimator. His work has been followed and extended by several authors. For instance, Noorbaloochi and Meeden (1983, 2000) expanded Lehmann's (1951) finding for a general class of prior distributions. Kass and Wasserman (1996) reviewed the problem of selecting prior distributions that their corresponding Bayes estimators are invariance under some sort of transforms. Meng and Zaslavsky (2002) considered a class of single observation unbiased priors (i.e. such priors produced unbiased Bayes estimator under squared error loss function). They showed that under mild regularity conditions, such class of priors must be "noninformative" for estimating either *location* or *scale* parameters. Gelman (2006) constructed a non-central t student family of conditionally conjugate priors for hierarchical standard deviation parameters. For restricted parameter space, Kucerovsky, Marchand, Payandeh, and Strawderman (2009) provided a class of prior distributions which their corresponding Bayes estimator under absolute value error loss equal to the maximum likelihood estimator. Ma and Leijon (2011) found a conjugate beta mixture prior such that its corresponding Bayes estimator under the variational inference framework retains some given properties.

This paper provides a class of prior distributions that their corresponding Bayes estimator under general convex and invariant loss function coincides with the MRE estimator for location or scale family of distributions.

Section 2 collects some required elements for other sections. The problem of finding such prior distribution for location and scale parameters have been studied in Sections 3 and 4, respectively.

2. Preliminaries

Bayes estimator for an unknown parameter θ under a general loss function ρ has been evaluated from the posterior distribution $\pi(\theta|x)$. Therefore, to study some specific properties of a Bayes estimator, one has to study posterior distribution $\pi(\theta|x)$. The following provides a condition which leads to equivalent Bayes estimator under two prior distributions.

Lemma 1 *Suppose X is a continuous random variable with density function f. Moreover, suppose that π_1 and π_2 are two priors distributions which lead to Bayes estimators δ_{π_1} and δ_{π_2}, under a general loss function ρ, respectively. Then, two Bayes estimators δ_{π_1} and δ_{π_2}, are equivalent estimator (i.e. $\delta_{\pi_1}(x) \equiv \delta_{\pi_2}(x)$) if and only if $\pi_1(\theta) = c\pi_2(\theta)$, for all $\theta \in \Theta$.*

Proof Bayes estimators with respect to π_1 and π_2 are equivalent if and only if the posterior distribution $\theta|x$ under these priors are equivalent, i.e.

$$\frac{\pi_1(\theta)f(x,\theta)}{\int_\Theta \pi_1(\theta)f(x,\theta)d\theta} = \frac{\pi_2(\theta)f(x,\theta)}{\int_\Theta \pi_2(\theta)f(x,\theta)d\theta}$$
$$\Leftrightarrow \frac{\pi_1(\theta)}{\pi_2(\theta)} = \frac{\int_\Theta \pi_1(\theta)f(x,\theta)d\theta}{\int_\Theta \pi_2(\theta)f(x,\theta)d\theta}$$

The rest of proof arrives from the fact that left hand side of the above equation is a function of θ while the right hand side is a function of x.

The following from Marchand and Payandeh (2011) recalls the Bayes estimator under generalized loss function ρ for location parameter μ.

Lemma 2 *(Marchand & Payandeh, 2011) Suppose random variable X sampled from a location density function g_0. Moreover, suppose that $\delta_\pi(x)$ stands for the Bayes estimator under generalized loss function ρ_1 and prior distribution $\pi(\mu)$ for location parameter μ. Then, $\delta_\pi(x)$ satisfies*

$$\int_{-\infty}^{\infty} \rho_1'(\delta_\pi(x) - \mu) g_0(x-\mu)\pi(\mu) d\mu = 0, \quad x \in R \tag{1}$$

Now, we extend the above result for the problem of finding a Bayes estimator for a scale parameter θ, under general loss function ρ_2 and prior distribution $\tau(\theta)$.

Lemma 3 *Suppose random variable Y sampled from a scale density function f_1. Moreover, suppose $\delta_\pi^*(y)$ stands for the Bayes estimator under generalized loss function ρ_2 and prior distribution $\tau(\theta)$ for a scale parameter θ. Then, $\delta_\pi^*(y)$ satisfies*

$$\int_{0}^{\infty} \rho_2'\left(\frac{\delta_\pi^*(y)}{\theta}\right) f_1\left(\frac{y}{\theta}\right) \frac{\pi(\ln(\theta))}{\theta^3} d\theta = 0, \quad y \geq 0 \tag{2}$$

where $\tau(\theta) = \pi(\ln(\theta))/\theta$.

Proof Marchand and Strawderman (2005) provided a connection between a scale parameter estimation problem, with elements (Y, θ, f_1, ρ_2) and a location parameter estimation problem with elements (Y, μ, g_0, ρ_1). They showed that by choosing $Y = ln(X)$, $\mu = \ln(\theta)$, $g_0(z) = e^z f_1(e^z)$, and $\rho_1(z) = \rho_2(e^z)$ transformations the problem of finding a Bayes estimator for a scale parameter θ under general loss function ρ_2 and prior distribution $\tau(\theta)$ can be restated as a problem of finding a Bayes estimator for location parameter μ under general loss function ρ_1 and prior distribution $\pi(\mu)$. Moreover, they showed that the Bayes estimator $\delta_\tau(x)$ of a location parameter μ with respect to prior $\tau(\mu)$ and the Bayes estimator $\delta_\pi^*(y)$ of a scale parameter θ with respect to prior $\pi(\theta)$ satisfies $\delta_\pi^*(y) = \exp\{\delta_\tau(\ln(x))\}$ and $\pi(\theta) = \tau(\ln(\theta))/\theta$. The desired proof arrives from the above observations.

The Fourier transform is an integral transform which defines for an integrable and real-valued $f: \mathbb{R} \to \mathbb{C}$ by

$$\mathcal{F}(f(t); t; \omega) := \int_{\mathbb{R}} f(t) e^{-i\omega t} dt, \quad \omega \in \mathbb{C} \tag{3}$$

The convolution theorem for the Fourier transforms states that

$$\mathcal{F}\left(\int_{\mathbb{R}} f(x-t) g(t) dt; x; \omega\right) = \mathcal{F}\left(f(x); x; \omega\right) \mathcal{F}\left(g(x); x; \omega\right) \tag{4}$$

(see Dym & Mckean, 1972 for more details).

Now, we recall definition and some properties of exponential type functions which are used later in proof of Theorem 4.

Definition 1 A function f in $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ is said to be an exponential type T function on the domain $D \subseteq \mathbb{C}$ if there are positive constants M and T such that $|f(\omega)| \leq M \exp\{T|\omega|\}$, for $\omega \in D$.

The Paley–Wiener theorem states that the Fourier (or inverse Fourier) transform of an $L_2(\mathbb{R})$ function vanishes outside of an interval $[-T, T]$, if and only if the function is an exponential type T (Dym & McKean, 1972, p. 158). An exponential type function is a continuous function which is infinitely differentiable everywhere and has a Taylor series expansion over every interval (see Champeney, 1987, p. 77; Walnut, 2002, p. 81). The exponential type functions are also called band-limited functions (see Bracewell, 2000, p. 119 for more details).

3. Bayesianity of the MRE estimator for location family of distributions

Suppose X has distribution P_μ with respect to Lebesgue density function $g_0(x-\mu)$ where $g_0(\cdot)$ is known and unknown location parameter $\mu \in \mathbb{R}$ is to be estimated by decision rule a under equivalent loss function $L(\mu, a) := \rho_1(a-\mu)$. An estimator (decision rule) T of μ is location invariant if and only if $T(X+c) = T(X)+c$, for all $c \in \mathbb{R}$. The MRE estimator is an estimator which has the smallest risk among all invariant estimators (Shao, 2003). It is well known that the MRE estimator for μ based upon one observation is $X+r$ where the value of r will minimize $E_0[l(X+r)]$, or for smooth $l(\cdot)$ satisfies $E_0[l'(X+r)] = 0$ (Lehmann & Casella, 1998).

Now, we consider the problem of finding prior distribution $\pi(\mu)$ that its corresponding Bayes estimator $\delta^{\rho_1}_{\text{Bayes}}(\cdot)$ under general convex and invariant loss function ρ_1, coincides with the MRE estimator $X+r$, i.e.

$$\delta^{\rho_1}_{\text{Bayes}}(X) \equiv X + r \tag{5}$$

The following studies possible solution of Equation 5 under the absolute value loss function.

THEOREM 1 *Suppose X is a continuous random variable with location density function g_0. Then, the Bayes estimator under the absolute value loss function and prior μ, coincides with the MRE estimator $X+r$, whenever $\pi(\mu) \equiv \text{constant}$, for $\mu \in \mathbb{R}$.*

Proof Under the absolute value loss function and prior π, $\delta_\pi(x) = x$ is a Bayes estimator if and only if

$$p(\mu \le x + r | X = x) = \frac{1}{2}$$
$$\int_{-\infty}^{x+r} \pi(\mu) g_0(x-\mu) d\mu = \int_{x+r}^{\infty} \pi(\mu) g_0(x-\mu) d\mu$$

Letting $\pi(\mu) = c + \pi^*(\mu)$, $k(x) = g_0(x) I_{(-\infty,r)}(x) - f_0(x) I_{(r,\infty)}(x)$, and $a = -c\int_{-\infty}^{\infty} k(x)dx$. The above equation can be restated as

$$\int_{-\infty}^{\infty} \pi^*(\mu) k(x-\theta) d\theta = a, \quad x \in R$$

An application of the Fourier transform along with the convolution theorem (Equation 4), then, taking the Fourier transform back lead to

$$\begin{aligned}\pi(\mu) &= c + \mathcal{F}^{-1}\left(\frac{2a\pi\text{Direct}(\omega)}{\mathcal{F}(k(x); x; \omega)}; \omega; \mu\right)\\ &= \text{constant}\end{aligned}$$

where Direct, $\mathcal{F}$, and $\mathcal{F}^{-1}$ stand for delta direct function and the Fourier and inverse Fourier transforms, respectively. The desired proof arrives from an application of Lemma 1.

The following theorem using Lemma 2 extends results of Theorem 1 to a general convex and invariant loss function ρ_1.

THEOREM 2 *Suppose X is a continuous random variable with location density function g_0. Moreover, suppose that $\rho_1(\cdot)$ stands for a convex and invariant loss function. Then, under prior $\pi(\mu) \equiv \text{constant}$, the MRE estimator $X+r$ is a Bayes estimator for a location parameter μ.*

Proof Lemma 2 states that the Bayes estimate $\delta_\pi(x)$ satisfies the equation $\int_{-\infty}^{\infty} \rho'(\delta_\pi(x)-\mu)\,g_0(x-\mu)\,\pi(\theta)d\mu = 0$, for all x. Setting $\pi(\mu) = c + \pi^*(\mu)$, one can reduce solving equation $\delta_\pi(x) \equiv x$, in π, to

$$\begin{aligned}
&\int_{-\infty}^{\infty} \rho'(\delta_\pi(x)-\mu)g_0(x-\mu)\pi^*(\mu)d\mu = -cb\\
&\quad\Leftrightarrow \mathcal{F}(\pi^*(\mu);\sim\mu;\sim\omega) = \frac{-2cb\pi\text{Direct}(\omega)}{\mathcal{F}(\rho'(-\mu-r)g_0(-\mu);\ \mu;\ \omega)}\\
&\quad\Leftrightarrow \pi(\mu) = c - 2cb\pi\mathcal{F}^{-1}\left(\frac{\text{Direct}(\omega)}{\mathcal{F}(\rho'(-\mu-r)g_0(-\mu);\mu;\omega)};\omega;\mu\right)\\
&\quad\pi(\mu) \equiv \text{constant}
\end{aligned}$$

where $b = \int_{-\infty}^{\infty} \rho'(t)g_0(t)dt$. The last equation arrives from that fact that there exist a delta direct function in the nominator of the function insides of the inverse Fourier transform. An application of Lemma 1 warranties that, if there is another prior distribution $\pi_1(\mu)$ such that its corresponding Bayes estimator $\delta_{\pi_1}(x) \equiv x$. Then, $\pi_1(\mu) = c\pi(\mu)$, for all $\mu \in \Theta$.

The following two propositions verify findings of Theorem 2 for two class of loss functions.

The MRE estimator for normal distribution under the LINEX loss function $\rho_{\text{LINEX}}(\delta,\mu) := exp\{a(\delta-\mu)\} - a(\delta-\mu) - 1$ is $x - a/2$. The following proposition studies Bayes ianity of such MRE estimator.

Proposition 1 Suppose X is a random variable which distributed according to a normal distribution with mean μ and variance 1. Then, the Bayes estimator for μ with respect to prior distribution $\pi(\mu) \equiv \text{constant}$, and under the LINEX loss function $\rho_{\text{LINEX}}(\delta,\mu) := \exp\{a(\delta-\mu)\} - a(\delta-\mu) - 1$ coincides with the MRE estimator $X - a/2$.

Proof The Bayes estimate, say $\delta_\pi(x)$, with respect to prior π and under loss function ρ_{LINEX} finds out by $\delta_\pi(x) = -\ln(E_\pi(\exp\{-a\mu\}|X=x))/a$. Setting $\pi(\mu) = c + \pi^*(\mu)$, Theorem 2 reduces solving equation $\delta_\pi(x) \equiv x - a/2$, in π, to

$$\begin{aligned}
\pi(\mu) &= c - \frac{ce^{a^2/2}\sqrt{8\pi^3}}{a}\mathcal{F}^{-1}\left(\frac{\text{Direct}(\omega)}{\mathcal{F}((e^{a(\mu-a/2)}-1)e^{-\mu^2/2};\ \mu;\ \omega)};\ \omega;\ \mu\right)\\
&= c - \frac{ce^{a^2/2}\sqrt{8\pi^3}}{a}\left(-\frac{\sqrt{2}}{8\sqrt{\pi^3}}\right)\\
&= c\left(1 + \frac{1}{2a}e^{a^2/2}\right)\\
&= \text{constant}
\end{aligned}$$

where constance c should be chosen such that $\pi(\mu) > 0$, for all $\mu \in \mathbb{R}$.

The following extends the above result to a convex combination of two LINEX loss functions, say ρ_{CLINEX}.

Proposition 2 Suppose X is a random variable which distributed according to a normal distribution with mean μ and variance 1. Then, the Bayes estimator for μ with respect to prior distribution $\pi(\mu) \equiv \text{constant}$, and under convex combination of two LINEX loss functions $\rho_{\text{CLINEX}}(\delta,\mu) := \alpha(\exp\{a(\delta-\mu)\} - a(\delta-\mu) - 1) + (1-\alpha)(\exp\{-a(\delta-\mu)\} + a(\delta-\mu) - 1)$ coincides with the MRE estimator $X + r$.

Proof Setting $\pi(\mu) = c + \pi^*(\mu)$, along with result of Theorem 2, one may reduce solving equation $\delta_\pi(x) \equiv x + r$, in π, to

$$\begin{aligned}\pi(\mu) &= c - \frac{cb\sqrt{8\pi^3}}{a}\mathcal{F}^{-1}\left(\frac{\text{Direct}(\omega)}{\mathcal{F}((\alpha(e^{a(\mu+r)}-1)-(1-\alpha)(e^{-a(\mu+r)}+1))e^{-\mu^2/2};\mu;\omega)};\omega;\mu\right)\\ &= c - \frac{cb\sqrt{8\pi^3}}{2\pi a}\left(\frac{2+2e^{a^2/2-ra}+4\alpha e^{a^2/2}\sinh(ra)-4\alpha}{1+e^{a^2/2-ra}+2\alpha e^{a^2/2}\sinh(ra)-2\alpha}\right)\\ &= \text{constant}\end{aligned}$$

where $b := \int_{-\infty}^{\infty}(\alpha(e^{at}-1)-(1-\alpha)(e^{-at}+1))e^{-t^2/2}/\sqrt{2\pi}dt$ and constance c should be chosen such that $\pi(\mu) > 0$, for all $\mu \in \mathbb{R}$. The second equality arrives from the fact that $\mathcal{F}(e^{-\mu^2/2};\mu;\omega) = \sqrt{2\pi}e^{-\omega^2/2}$, $\mathcal{F}\left(e^{\pm a(\mu+r)-\mu^2/2};\mu;\omega\right) = \sqrt{2\pi}e^{\pm ar-(\omega\pm ai)^2/2}$, and $\mathcal{F}^{-1}(\text{Direct}(\omega)h(\omega);\omega;\mu) = h(0)$.

4. Bayesianity of the MRE estimator for scale family of distributions

Suppose X has distribution P_θ with respect to Lebesgue density function $f_1(x/\theta)$ where $f_1(\cdot)$ is known and unknown scale parameter $\theta \in \mathbb{R}^+$ is to be estimated by decision rule a under equivalent loss function $L(\theta, a) := \rho_2(a/\theta)$. An estimator (decision rule) T of θ is scale invariant if and only if $T(cX) = cT(X)$, for all $c \in \mathbb{R}$. The MRE estimator is an estimator which has the smallest risk among all invariant estimators (Shao, 2003). It is well known that the MRE estimator for θ based upon one observation is rX where the value of r will minimize $E_1[l(rX)]$, for smooth $l(\cdot)$ satisfies $E_1[l'(rX)] = 0$ (Lehmann & Casella, 1998).

Now, we consider the problem of finding prior distribution $\pi(\theta)$ that its corresponding Bayes estimator $\delta^{\rho_2}_{\text{Bayes}}(\cdot)$ under general convex and invariant loss function ρ_2, coincides with the MRE estimator rX, i.e.

$$\delta^{\rho_2}_{\text{Bayes}}(X) \equiv rX \tag{6}$$

The following studies possible solution of Equation 6 under the absolute value loss function for symmetric-scale distribution functions (see Jafarpour & Farnoosh, 2005 for more details on symmetric-scale distribution functions).

THEOREM 3 *Suppose non-negative and continuous random variable X distributed according to a scale density function f_1. Moreover, suppose that X/θ is symmetric about $1/r$. Then, the Bayes estimator, under the absolute value loss function and prior distribution $\tau(\theta) = 1/\theta$, coincides with the MRE estimator rX.*

Proof Under the absolute value loss function, solving equation $\delta_\tau(x) \equiv rx$, in τ, can be restated as

$$\begin{aligned} median(\theta|X=x) = rx &\Leftrightarrow \int_0^{rx}\frac{\tau(\theta)}{\theta}f_1\left(\frac{x}{\theta}\right)d\theta = \int_{rx}^{\infty}\frac{\tau(\theta)}{\theta}f_1\left(\frac{x}{\theta}\right)d\theta\\ (\text{let } \ y = x/\theta) &\Leftrightarrow \int_0^{1/r}\frac{\tau(x/y)}{y}f_1(y)dy = \int_{1/r}^{\infty}\frac{\tau(x/y)}{y}f_1(y)dy\\ &\Leftrightarrow 2E_{f_1}\left(\frac{\tau(x/Y)}{Y}I_{[0,1/r]}(Y)|X=x\right) - E_{f_1}\left(\frac{\tau(x/Y)}{Y}|X=x\right) = 0\\ &\Leftrightarrow Cov_{f_1}\left(I_{[0,1/r]}(Y), \frac{\tau(x/Y)}{Y}|X=x\right) = 0\end{aligned}$$

From the above equation, one may conclude that, a trivial solution of τ is to let $\tau(x/Y)/Y$ be free of Y. This observation along with an application of Lemma 1 complete the desired proof.

$X \sim Unif(0, 2\theta/r), \theta,\ r > 0$, is an obvious example for such symmetric-scale distribution satisfies Theorem 3's conditions. Since $rX/\theta \sim Unif(0, 2)$ an expression $E(\rho_2(rX/\theta))$ does not depend on θ. Therefore, rX is a MRE estimator (see Rohatgi & Saleh, 2011, p. 446 for more details).

The following theorem using Lemma 3 extends results of Theorem 3 to a general convex and invariant loss function ρ_2.

THEOREM 4 *Suppose non-negative and continuous random variable X distributed according to a scale family distribution with density function f_1. Then, Bayes estimator, under convex and invariant loss function ρ_2 and prior distribution*

$$\tau(\theta) = \frac{c}{\theta}\left(1 - \mathcal{F}^{-1}\left(\frac{b\sqrt{\pi}e^{-(\omega-2)^2/4}}{\mathcal{F}(\rho_2'(re^{-y})f_1(e^{-y});\, y;\, \omega-2)};\omega;\,\ln(\theta)\right)\right)$$

coincides with the MRE estimator rX.

Proof Using Lemma 3, one may conclude that solving equation $\delta_\tau(x) \equiv rx$, in π^*, where $\tau(\theta) = (c + \pi^*(\ln(\theta)))/\theta$, can be reduced to

$$-c\frac{b}{x^2} = \int_0^\infty \rho_2'\left(\frac{rx}{\theta}\right) f_1\left(\frac{x}{\theta}\right)\frac{\pi^*(\ln(\theta))}{\theta^3}d\theta$$
$$-cbe^{-z^2} = \int_{-\infty}^\infty \rho_2'(e^{z-\gamma+\ln(r)})f_1(e^{z-\gamma})\pi^*(\gamma)e^{-2\gamma}d\gamma$$

where $b = \int_0^\infty t\rho_2'(rt)f_1(t)dt$ and in the second equality $z = \ln(x)$ and $\gamma = \ln(\theta)$. Taking the Fourier transform from both sides along with an application of the convolution theorem, the above equation can be restated as

$$\begin{aligned}
\mathcal{F}(\pi^*(\gamma)e^{-2\gamma};\gamma;\omega) &= \frac{-cb\sqrt{\pi}e^{-\omega^2/4}}{\mathcal{F}(\rho_2'(re^{-y})f_1(e^{-y});\, y;\, \omega)}\\
\Leftrightarrow \pi^*(\gamma) &= -ce^{2\gamma}\mathcal{F}^{-1}\left(\frac{b\sqrt{\pi}e^{-\omega^2/4}}{\mathcal{F}(\rho_2'(re^{-y})f_1(e^{-y});\, y;\, \omega)};\,\omega;\,\gamma\right)\\
\Leftrightarrow \pi^*(\gamma) &= -c\mathcal{F}^{-1}\left(\frac{b\sqrt{\pi}e^{-(\omega-2)^2/4}}{\mathcal{F}(\rho_2'(re^{-y})f_1(e^{-y});\, y;\, \omega-2)};\,\omega;\,\gamma\right)\\
\Leftrightarrow \pi^*(\ln(\theta)) &= -c\mathcal{F}^{-1}\left(\frac{b\sqrt{\pi}e^{-(\omega-2)^2/4}}{\mathcal{F}(\rho_2'(re^{-y})f_1(e^{-y});\, y;\, \omega-2)};\,\omega;\,\ln(\theta)\right)\\
\Leftrightarrow \tau(\theta) &= \tfrac{c}{\theta}\left(1-\mathcal{F}^{-1}\left(\frac{b\sqrt{\pi}e^{-(\omega-2)^2/4}}{\mathcal{F}(\rho_2'(re^{-y})f_1(e^{-y});\, y;\, \omega-2)};\,\omega;\,\ln(\theta)\right)\right)
\end{aligned}$$

Positivity of $\tau(\cdot)$ arrives from the fact that $b\sqrt{\pi}e^{-(\omega-2)^2/4}/\mathcal{F}(\rho_2'(re^{-y})f_1(e^{-y});y;\omega-2)$ is an exponential type 1 function. Now, the Paley–Wiener theorem warranties that its corresponding inverse Fourier transform takes its values inside of an interval $[-1,1]$. The desired proof arrives by an application of Lemma 1.

It worthwhile mentioning that the above inverse Fourier transform may not evaluated analytically. Therefore, one has to employ some numerical approach, such as the fast Fourier transformation, to handel it.

5. Conclusion and suggestion

This paper provides a class of prior distributions which their corresponding Bayes estimator under general convex and invariant loss function coincides with the MRE estimator for location or scale family of distributions. This problem can be studied for scale-location family of distributions under general convex and invariant loss function.

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