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On analytical and numerical study of implicit fixed point iterations

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Abstract: In this article, we define a new three-step implicit iteration and study its strong convergence, stability and data dependence. It is shown that the new three-step iteration has better rate of convergence than implicit and explicit Mann iterations as well as implicit Ishikawa-type iteration. Numerical example in support of validity of our results is provided.

Subjects: Science; Mathematics & Statistics; Applied Mathematics

Keywords: implicit iterations; strong convergence; convergence rate; stability; data dependence

AMS subject classifications: 47H10; 47H17; 47H15

1. Introduction

Implicit iterations are of great importance from numerical standpoint as they provide accurate approximation compared to explicit iterations. Computer-oriented programmes for the approximation of fixed point by using implicit iterations can reduce the computational cost of the fixed-point problem. Numerous papers have been published on convergence of explicit as well as implicit iterations in various spaces (Anh & Binh, 2004; Berinde, 2004; Berinde, 2011; Chidume & Shahzad, 2005; Chugh & Kumar, 2013; Ciric, Rafiq, Cakić, & Ume, 2009; Ćirić, Rafiq, Radenović, Rajović, & Ume, 2008; Khan, Fukhar-ud-din, & Khan, 2012; Rhoades, 1993; Shahzad & Zegeye, 2009). Data dependence of fixed points is a related and new issue which has been studied by many authors; see (Gursoy,

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PUBLIC INTEREST STATEMENT

In this paper, a new three-step implicit iteration is defined as:

$$x_n = W(x_{n-1}, Ty_n, \alpha_n)$$
$$y_n = W(z_n, Tz_n, \beta_n)$$
$$z_n = W(x_n, Tx_n, \gamma_n)$$

where α_n, β_n and γ_n are sequences in [0, 1].

For this three-step implicit iteration, strong convergence and stability results are proved in convex metric spaces. Also, we have done the comparison of rate of convergence of newly defined iteration with implicit and explicit Mann, and implicit and explicit Ishikawa-type iterations analytically and numerically. It is found that our newly defined implicit iteration has better rate of convergence than these iterations. Also, data-dependence result for new implicit iteration is proved in hyperbolic spaces.

Karakaya, & Rhoades, 2013; Khan, Kumar, & Hussain, 2014 and references therein). In computational mathematics, it is of theoretical and practical importance to compare the convergence rate of iterations and to find out, if possible, which one of them converges more rapidly to the fixed point. Recent works in this direction are (Chugh & Kumar, 2013; Ciric, Lee, & Rafiq, 2010; Hussian, Chugh, Kumar, & Rafiq, 2012; Khan et al., 2014; Kumar et al., 2013). Motivated by the works of Ciric (Ciric, 1971; Ciric, 1974; Ciric, 1977; Ciric et al., 2010; Ćirić & Nikolić, 2008a, 2008b; Ciric et al., 2009; Ćirić et al., 2008; Ciric, Ume, & Khan, 2003) and the fact that three-step iterations give better approximation than one-step and two-step iterations (Glowinski & Tallec, 1989), we define a new and more general three-step implicit iteration with higher convergence rate as compared to implicit Mann, explicit Mann and implicit Ishikawa iterations.

Let K be a nonempty convex subset of a convex metric space X and $T{:}K \to K$ be a given mapping. Then for $x_0 \in K$, we define the following implicit iteration:

$$\begin{aligned} x_n &= W(x_{n-1}, Ty_n, \alpha_n) \\ y_n &= W(z_n, Tz_n, \beta_n) \\ z_n &= W(x_n, Tx_n, \gamma_n) \end{aligned} \tag{1.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$;$\{\gamma_n\}$ are sequences in [0, 1].

Equivalence form of iteration (1.1) in linear space can be written as

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)Ty_n \\ y_n &= \beta_n z_n + (1-\beta_n)Tz_n \\ z_n &= \gamma_n x_n + (1-\gamma_n)Tx_n \end{aligned} \tag{IN}$$

Putting $\gamma_n = 1$ in (IN), we get Ishikawa-type implicit iteration:

$$\begin{aligned} x_n &= \alpha_n x_{n-1} + (1-\alpha_n)Ty_n \\ y_n &= \beta_n x_n + (1-\beta_n)Tx_n \end{aligned} \tag{II}$$

Putting $\gamma_n = \beta_n = 1$ in (IN), we get well-known implicit Mann iteration (Ćirić et al., 2008; Ciric et al., 2003):

$$x_n = W(x_{n-1}, Tx_n, \alpha_n) = \alpha_n x_{n-1} + (1-\alpha_n)Tx_n \tag{IM}$$

Also, Mann iteration (Mann, 1953) is defined as :

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n Tx_n \tag{M}$$

Zamfirescu operators (Zamfirescu, 1972) are most general contractive-like operators which have been studied by several authors, satisfying the following condition: for each pair of points x, y in X, at least one of the following is true:

$$\begin{aligned} &\text{(i)} \quad d(Tx, Ty) \le pd(x, y) \\ &\text{(ii)} \quad d(Tx, Ty) \le q[d(x, Tx) + d(y, Ty)] \\ &\text{(iii)} \quad d(Tx, Ty) \le r[d(x, Ty) + d(y, Tx)] \end{aligned} \tag{1.2}$$

where p, q, r are nonnegative constants satisfying $0 \le p \le 1$, $0 \le q, r \le \frac{1}{2}$.

Z-operators are equivalent to the following contractive contraction:

$$\begin{aligned} &d(Tx, Ty) \le c\max\left\{d(x, y), \left\{d(x, Tx) + d(y, Ty)\right\}/2, \left\{d(x, Ty) + d(y, Tx)\right\}/2\right\} \\ &\quad \forall x, y \in X, 0 < c < 1 \end{aligned} \tag{1.3}$$

The contractive condition (1.3) implies

$$d\,(Tx, Ty) \le 2ad\,(x, Tx) + ad\,(x, y)\,, \forall x, y \in X \tag{1.4}$$

where $a = \max\left\{c, \frac{c}{2-c}\right\}$ (see Berinde, 2004).

Rhoades (1993) used the following more general contractive condition than (1.4): there exists $c \in [0, 1)$, such that

$$d(Tx, Ty) \le c\max\left\{d(x,y), \left\{d(x,Tx) + d(y,Ty)\right\}/2, d(x,Ty), d(y,Tx)\right\} \quad \forall x, y \in X \tag{1.5}$$

Osilike (1995) used a more general contractive definition than those of Rhoades': there exists $a \in (0, 1)$, $L \ge 0$, such that

$$d\,(Tx, Ty) \le Ld\,(x, Tx) + ad\,(x, y)\,\forall x, y \in X \tag{1.6}$$

We use the contractive condition due to Imoru and Olatinwo (Olatinwo & Imoru, 2008), which is more general than (1.6): there exists $a \in [0, 1)$ and a monotone-increasing function $\varphi: R^+ \to R^+$ with $\phi(0) = 0$, such that

$$d(Tx, Ty) \le \phi(d(x, Tx)) + ad(x, y), \quad a \in [0, 1), \forall x, y \in X \tag{1.7}$$

Also, we use the following definitions and lemmas to achieve our main results.

Definition 1.1 (Takahashi, 1970) A map $W{:}X^2 \times [0, 1] \to X$ is a convex structure on X if

$$d(u, W(x, y, \lambda)) \le \lambda d(u, x) + (1 - \lambda)d(u, y)$$

for all $x, y, u \in X$ and $\lambda \in [0, 1]$. A metric space (X, d) together with a convex structure W is known as convex metric space and denoted by (X, d, W). A nonempty subset C of a convex metric space is convex if $W(x, y, \lambda) \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$.

All normed spaces and their subsets are the examples of convex metric spaces. But there are many examples of convex metric spaces which are not embedded in any normed space (see Takahashi, 1970). Several authors extended this concept in many ways later, one such convex structure is hyperbolic space introduced by Kohlenbach (2004) as follows:

Definition 1.2 (Kohlenbach, 2004) A hyperbolic space (X, d, W) is a metric space (X, d) together with a convexity mapping $W: X^2 \times [0, 1] \to X$ satisfying

(W1) $d(z, W(x, y, \lambda)) \le (1 - \lambda)\, d(z, x) + \lambda\, d(z, y)$

(W2) $d(W(x, y, \lambda_1), W(x, y, \lambda_2)) = |\lambda_1 - \lambda_2|\, d(x, y)$

(W3) $W(x, y, \lambda) = W(y, x, 1 - \lambda)$

(W4) $d(W(x, z, \lambda), W(y, w, \lambda)) \le (1 - \lambda)d(x, y) + \lambda d(z, w)$ for all $x, y, z, w \in X$ and $\lambda, \lambda_1, \lambda_2 \in [0, 1]$.

Evidently, every hyperbolic space is a convex metric space but converse may not true. For example, if $X = R$, $W(x, y, \lambda) = \lambda\, x + (1 - \lambda)\, y$ and define $d\,(x, y) = \frac{|x-y|}{1+|x-y|}$ for $x, y \in R$, then (X, d, W) is a convex metric space but not a hyperbolic space.

The stability of explicit as well as implicit iterations has extensively been studied by various authors (Berinde, 2011; Khan et al., 2014; Olatinwo, 2011; Olatinwo & Imoru, 2008; Ostrowski, 1967; Timis, 2012) due to its increasing importance in computational mathematics, especially due to revolution in computer programming. The concept of T-stability in convex metric space setting was given by Olatinwo (Olatinwo, 2011):

Definition 1.3. (Olatinwo, 2011) Let (X, d, W) be a convex metric space and $T{:}X \to X$ a self-mapping.

Let $\{x_n\}_{n=0}^{\infty} \subset X$ be the sequence generated by an iterative scheme involving T, which is defined by

$$x_{n+1} = f_{T,\alpha_n}^{x_n}, \quad n = 0, 1, 2, \ldots \tag{1.8}$$

where $x_0 \in X$ is the initial approximation and $f_{T,\alpha_n}^{x_n}$ is some function having convex structure, such that $\alpha_n \in [0, 1]$. Suppose that $\{x_n\}$ converges to a fixed-point p of T. Let $\{y_n\}_{n=0}^{\infty} \subset X$ be an arbitrary sequence and set $\varepsilon_n = d(y_{n+1}, f_{T,\alpha_n}^{y_n})$. Then, the iteration (1.8) is said to be T-stable with respect to T if and only if $\lim_{n\to\infty} \varepsilon_n = 0$, implies $\lim_{n\to\infty} y_n = p$.

Lemma 1.4 *(Berinde, 2004; Khan et al., 2014) If δ is a real number such that $0 \leq \delta < 1$ and $\{\in_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \in_n = 0$, then for any sequence of positive numbers $\{u_n\}_{n=0}^{\infty}$ satisfying*

$$u_{n+1} \leq \delta u_n + \in_n, n = 0, 1, 2, \ldots$$

we have $\lim_{n\to\infty} u_n = 0$.

Definition 1.5 (Berinde, 2004) Suppose $\{a_n\}$ and $\{b_n\}$ are two real convergent sequences with limits a and b, respectively. Then $\{a_n\}$ is said to converge faster than $\{b_n\}$ if

$$\lim_{n\to\infty} \left| \frac{a_n - a}{b_n - b} \right| = 0$$

Definition 1.6 (Berinde, 2004) Let $\{u_n\}$ and $\{v_n\}$ be two fixed-point iterations that converge to the same fixed point p on a normed space X, such that the error estimates

$$\|u_n - p\| \leq a_n$$

and

$$\|v_n - p\| \leq b_n$$

are available, where $\{a_n\}$ and $\{b_n\}$ are two sequences of positive numbers (converging to zero) .If $\{a_n\}$ converge faster than $\{b_n\}$, then we say that $\{u_n\}$ converge faster to p than $\{v_n\}$.

Definition 1.7 (Gursoy et al., 2013) Let T, T_1 be two operators on X. We say T_1 is approximate operator of T if for all $x \in X$ and for a fixed $\in > 0$, we have $d(Tx, T_1x) \leq \in$.

Lemma 1.8 *(Gursoy et al., 2013; Khan et al., 2014) Let $\{a_n\}_{n=0}^{\infty}$ be a nonnegative sequence for which there exists $n_0 \in N$, such that for all $n \geq n_0$, one has the following inequality:*

$$a_{n+1} \leq (1 - r_n)\, a_n + r_n t_n$$

where $r_n \in (0, 1)$, for all $n \in N$, $\sum_{n=1}^{\infty} r_n = \infty$ and $t_n \geq 0 \ \ \forall n \in N$.

Then, $0 \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} t_n$.

Having introduced the implicit iteration (1.1), we use it to prove the results concerning convergence, stability and rate of convergence for contractive condition (1.7) in convex metric spaces. Furthermore, data-dependence result of the same iteration is proved in hyperbolic spaces.

2. Convergence and stability results for new implicit iteration in convex metric spaces

THEOREM 2.1 *Let K be a nonempty closed convex subset of a convex metric space X and T be a quasi-contractive operator satisfying (1.7) with $F(T) \neq \varphi$. Then, for $x_0 \in C$, the sequence $\{x_n\}$ defined by (1.1) with $\sum (1-\alpha_n) = \infty$, converges to the fixed point of T.*

Proof Using (1.1) and (1.7), we have for $p \in F(T)$,

$$\begin{aligned} d(x_n,p) &= d(W(x_{n-1},Ty_n,\alpha_n),p) \\ &\le \alpha_n d(x_{n-1},p) + (1-\alpha_n)d(Ty_n,p) \\ &\le \alpha_n d(x_{n-1},p) + (1-\alpha_n)ad(y_n,p) \end{aligned} \tag{2.1}$$

Now, we have the following estimates:

$$\begin{aligned} d(y_n,p) &= d(W(z_n,Tz_n,\beta_n),p) \\ &\le \beta_n d(z_n,p) + (1-\beta_n)d(Tz_n,p) \\ &\le \beta_n d(z_n,p) + (1-\beta_n)ad(z_n,p) \\ &= [\beta_n + a(1-\beta_n)]d(z_n,p) \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} d(z_n,p) &= d(W(x_n,Tx_n,\gamma_n),p) \\ &\le \gamma_n d(x_n,p) + (1-\gamma_n)d(Tx_n,p) \\ &\le \gamma_n d(x_n,p) + (1-\gamma_n)ad(x_n,p) \\ &= [\gamma_n + a(1-\gamma_n)]d(x_n,p) \end{aligned} \tag{2.3}$$

Inequalities (2.1), (2.2) and (2.3) yield

$$d(x_n,p) \le \alpha_n d(x_{n-1},p) + (1-\alpha_n)a[\beta_n + a(1-\beta_n)][\gamma_n + a(1-\gamma_n)]d(x_n,p)$$

which further implies

$$\{1-(1-\alpha_n)a[\beta_n + a(1-\beta_n)][\gamma_n + a(1-\gamma_n)]\}d(x_n,p) \le \alpha_n\, d(x_{n-1},p)$$

and therefore

$$d(x_n,p) \le \frac{\alpha_n}{1-(1-\alpha_n)a[\beta_n + a(1-\beta_n)][\gamma_n + a(1-\gamma_n)]} d(x_{n-1},p) \tag{2.4}$$

Let $\frac{P_n}{Q_n} = \frac{\alpha_n}{1-(1-\alpha_n)a[\beta_n+a(1-\beta_n)][\gamma_n+a(1-\gamma_n)]}$

then

$$\begin{aligned} 1-\frac{P_n}{Q_n} &= \frac{1-(1-\alpha_n)a[\beta_n+a(1-\beta_n)][\gamma_n+a(1-\gamma_n)]-\alpha_n}{1-(1-\alpha_n)a[\beta_n+a(1-\beta_n)][\gamma_n+a(1-\gamma_n)]} \\ &\ge 1-(1-\alpha_n)a[\beta_n + a(1-\beta_n)][\gamma_n + a(1-\gamma_n)] + \alpha_n \end{aligned}$$

which further implies,

$$\begin{aligned} \frac{P_n}{Q_n} &\le (1-\alpha_n)a[\beta_n + a(1-\beta_n)][\gamma_n + a(1-\gamma_n)] + \alpha_n \\ &= (1-\alpha_n)a[\beta_n + a(1-\beta_n)][\gamma_n + a(1-\gamma_n)] + \alpha_n \\ &= (1-\alpha_n)a[1-(1-a)(1-\beta_n)][1-(1-a)(1-\gamma_n)] + \alpha_n \end{aligned} \tag{2.5}$$

$$\le (1-\alpha_n)a + \alpha_n$$

$$= 1-(1-\alpha_n)(1-a). \tag{2.6}$$

Using (2.6), (2.4) becomes

$$d(x_n, p) \le [1-(1-\alpha_n)(1-a)]d(x_{n-1}, p)$$

$$\le \prod_{i=1}^{n}[1-(1-\alpha_i)(1-a)]d(x_0, p)$$

$$\le e^{-\sum_{i=1}^{n}(1-\alpha_i)(1-a)}d(x_0, p) \tag{2.7}$$

But $\sum_{i=1}^{n}(1-\alpha_i) = \infty$, hence (2.7) yields $\lim_{n\to\infty} d(x_n, p) = 0$. Therefore, $\{x_n\}$ converges to p.

THEOREM 2.2 *Let K be a nonempty closed convex subset of a convex metric space X and T be a quasi-contractive operator satisfying (1.7) with $F(T) \neq \varphi$. Then, for $x_0 \in C$, the sequence $\{x_n\}$ defined by (1.1) with $\alpha_n \le \alpha < 1$, $\sum(1-\alpha_n) = \infty$, is T-stable.*

Proof Suppose that $\{p_n\}_{n=0}^{\infty} \subset K$ be an arbitrary sequence, $\varepsilon_n = d(p_n, W(p_{n-1}, Tq_n, \alpha_n))$, where $q_n = W(r_n, Tr_n, \beta_n)$, $r_n = W(p_n, Tp_n, \gamma_n)$ and let $\lim_{n\to\infty} \varepsilon_n = 0$.

Then, using (1.7), we have

$$\begin{aligned} d(p_n, p) &\le d(p_n, W(p_{n-1}, Tq_n, \alpha_n)) + d(W(p_{n-1}, Tq_n, \alpha_n), p) \\ &\le \varepsilon_n + \alpha_n d(p_{n-1}, p) + (1-\alpha_n)d(Tq_n, p) \\ &\le \varepsilon_n + \alpha_n d(p_{n-1}, p) + (1-\alpha_n)\varphi d(Tp, p) + (1-\alpha_n)a d(q_n, p) \\ &\le \varepsilon_n + \alpha_n d(p_{n-1}, p) + (1-\alpha_n)a[\beta_n + a(1-\beta_n)][\gamma_n + a(1-\gamma_n)]d(p_n, p) \end{aligned} \tag{2.8}$$

which implies

$$\{1-(1-\alpha_n)a[\beta_n + a(1-\beta_n)][\gamma_n + a(1-\gamma_n)]\}d(p_n, p) \le \varepsilon_n + \alpha_n d(p_{n-1}, p)$$

and therefore

$$\begin{aligned} d(p_n, p) \le{}& \frac{\alpha_n}{1-(1-\alpha_n)a[\beta_n+a(1-\beta_n)][\gamma_n+a(1-\gamma_n)]}d(p_{n-1}, p) \\ &+ \frac{\varepsilon_n}{1-(1-\alpha_n)a[\beta_n+a(1-\beta_n)][\gamma_n+a(1-\gamma_n)]} \end{aligned} \tag{2.9}$$

But from (2.6), we have

$$\frac{\alpha_n}{1-(1-\alpha_n)a[\beta_n + a(1-\beta_n)][\gamma_n + a(1-\gamma_n)]} \le 1-(1-\alpha_n)(1-a) \tag{2.10}$$

Hence (2.9) becomes

$$d(p_n, p) \le [1-(1-\alpha_n)(1-a)]d(p_{n-1}, p) + \frac{\varepsilon_n}{1-(1-\alpha_n)a[\beta_n + a(1-\beta_n)][\gamma_n + a(1-\gamma_n)]} \tag{2.11}$$

Using $\alpha_n \le \alpha < 1$ and $a \in (0, 1)$,we have

$$1-(1-\alpha_n)(1-a) < 1$$

Hence, using Lemma 1.4, (2.11) yields $\lim_{n\to\infty} p_n = p$

Conversely, if we let $\lim_{n\to\infty} p_n = p$ then using contractive condition (1.7), it is easy to see that $\lim_{n\to\infty} \varepsilon_n = 0$.

Therefore, the iteration (1.1) is T-stable.

Remark 2.3 As contractive condition (1.7) is more general than those of (1.2)–(1.6), the convergence and stability results for implicit iteration (IN) using contractive conditions (1.2)–(1.6) can be obtained as special cases.

Remark 2.4 As implicit Mann iteration (IM) and Ishikawa-type iteration (II) are special cases of new implicit iteration (1.1), results similar to Theorem 2.1 and Theorem 2.2 hold for implicit Mann iteration (IM) and Ishikawa-type iteration (II).

3. Rate of convergence for implicit iterations

THEOREM 3.1 *Let K be a nonempty closed convex subset of a convex metric space X and T be a quasi-contractive operators satisfying (1.7) with $F(T) \neq \varphi$. Then, for $x_0 \in C$, the sequence $\{x_n\}$ defined by (1.1) with $\sum (1-\alpha_n) = \infty$, converges faster than implicit Mann iteration (IM) as well as Ishikawa-type iteration (II) to the fixed-point of T.*

Proof For implicit Mann iteration (IM), we have

$$\begin{aligned} d(x_n, p) &\leq \alpha_n d(x_{n-1}, p) + (1-\alpha_n) d(Tx_n, p) \\ &\leq \alpha_n d(x_{n-1}, p) + (1-\alpha_n) a d(x_n, p) \end{aligned}$$

which further yield

$$[1-(1-\alpha_n)a] d(x_n, p) \leq \alpha_n d(x_{n-1}, p)$$

and so

$$d(x_n, p) \leq \frac{\alpha_n}{1-(1-\alpha_n)a} d(x_{n-1}, p) \tag{3.1}$$

If we take $\frac{\alpha_n}{1-(1-\alpha_n)a} = \frac{A_n}{B_n}$

then,

$$1 - \frac{A_n}{B_n} = 1 - \frac{\alpha_n}{1-(1-\alpha_n)a} = \frac{1-[(1-\alpha_n)a+\alpha_n]}{1-(1-\alpha_n)a} \geq 1-[(1-\alpha_n)a+\alpha_n]$$

and hence

$$\frac{A_n}{B_n} \leq (1-\alpha_n)a + \alpha_n \tag{3.2}$$

Keeping in mind the Berinde's Definition 1.6, inequalities (2.6) and (3.3) yields fast convergence of three-step implicit iteration (IN) than implicit Mann iteration (IM).

Also, for explicit Mann iteration, we have

$$\begin{aligned} d(x_n, p) = d(W(x_{n-1}, Tx_{n-1}, \alpha_n), p) &\leq \alpha_n d(x_{n-1}, p) + (1-\alpha_n) d(Tx_{n-1}, p) \\ &\leq \alpha_n d(x_{n-1}, p) + (1-\alpha_n) a d(x_{n-1}, p) \\ &\leq [\alpha_n + (1-\alpha_n)a] d(x_{n-1}, p) \end{aligned} \tag{3.3}$$

Similarly, for implicit Ishikawa-type iteration (II), we have

$$d(x_n, p) \leq \{(1-\alpha_n)a[1-(1-a)(1-\beta_n)] + \alpha_n\} d(x_{n-1}, p) \tag{3.4}$$

Using (3.1), (3.2) and (3.3), we conclude that implicit Mann iteration converges faster than corresponding explicit Mann iteration. Also, from (2.5) and (3.4), it is obvious that new three-step implicit iteration converges faster than Ishikawa-type implicit iteration (II).

Example 3.2 Let $K = [0, 1]$, $T(x) = \frac{x}{4}$, $x \neq 0$ and $\alpha_n = \beta_n = \gamma_n = 1 - \frac{4}{\sqrt{n}}$, $n \geq 25$ and for $n = 1, 2, \ldots, 24$, $\alpha_n = \beta_n = \gamma_n = 0$, then for implicit Mann iteration, we have

$$
\begin{aligned}
x_n &= \alpha_n x_{n-1} + (1-\alpha_n) T x_n \\
&= \left(1 - \tfrac{4}{\sqrt{n}}\right) x_{n-1} + \tfrac{4}{\sqrt{n}} \tfrac{x_n}{4}
\end{aligned}
$$

which further implies

$$
x_n \left[1 - \frac{1}{\sqrt{n}}\right] = \left(1 - \frac{4}{\sqrt{n}}\right) x_{n-1}
$$

and so

$$
x_n = \frac{\sqrt{n}-4}{\sqrt{n}-1} x_{n-1} = \prod_{i=25}^{n} \left(\frac{\sqrt{i}-4}{\sqrt{i}-1}\right) x_0 \tag{3.5}
$$

Also, for the new three-step iteration (IN), we have

$$
\begin{aligned}
z_n &= \alpha_n x_n + (1-\alpha_n) T x_n \\
&= \left(1 - \tfrac{4}{\sqrt{n}}\right) x_n + \tfrac{4}{\sqrt{n}} \tfrac{x_n}{4} \\
&= \left(1 - \tfrac{3}{\sqrt{n}}\right) x_n
\end{aligned}
$$

$$
y_n = \left(1 - \frac{3}{\sqrt{n}}\right) z_n = \left(1 - \frac{3}{\sqrt{n}}\right)^2 x_n
$$

and so

$$
\begin{aligned}
x_n &= \left(1 - \tfrac{4}{\sqrt{n}}\right) x_{n-1} + \tfrac{4}{\sqrt{n}} \tfrac{y_n}{4} \\
&= \left(1 - \tfrac{4}{\sqrt{n}}\right) x_{n-1} + \tfrac{1}{\sqrt{n}} \left(1 - \tfrac{3}{\sqrt{n}}\right)^2 x_n \\
&= \left(1 - \tfrac{4}{\sqrt{n}}\right) x_{n-1} + \left(\tfrac{1}{\sqrt{n}} + \tfrac{9}{n^{3/2}} - \tfrac{6}{n}\right) x_n
\end{aligned}
$$

which further implies

$$
x_n \left[1 - \left(\frac{1}{\sqrt{n}} + \frac{9}{n^{3/2}} - \frac{6}{n}\right)\right] = \frac{\sqrt{n}-4}{\sqrt{n}} x_{n-1}
$$

and hence

$$
x_n = \frac{n^{3/2} - 4n}{n^{3/2} - n + 6\sqrt{n} - 9} x_{n-1}
$$

$$
= \prod_{i=25}^{n} \left(\frac{i^{3/2} - 4i}{i^{3/2} - i + 6\sqrt{i} - 9}\right) x_0 \tag{3.6}
$$

Also, for explicit Mann iteration, we have

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)Tx_{n-1} = \left(1 - \frac{4}{\sqrt{n}}\right)x_{n-1} + \frac{4}{\sqrt{n}}\frac{x_{n-1}}{4} = \left(1 - \frac{3}{\sqrt{n}}\right)x_{n-1} \tag{3.7}$$

For two-step Ishikawa-type implicit iteration, we have

$$y_n = \left(1 - \frac{3}{\sqrt{n}}\right)x_n$$

and so

$$x_n = \left(1 - \frac{4}{\sqrt{n}}\right)x_{n-1} + \left(\frac{1}{\sqrt{n}}\right)\left(1 - \frac{3}{\sqrt{n}}\right)x_n$$

$$x_n\left(1 - \frac{1}{\sqrt{n}} + \frac{3}{n}\right) = \left(1 - \frac{4}{\sqrt{n}}\right)x_{n-1}$$

$$x_n = \frac{(\sqrt{n}-4)\sqrt{n}}{n-\sqrt{n}+3}x_{n-1} = \frac{n-4\sqrt{n}}{n-\sqrt{n}+3}x_{n-1} = \prod_{i=25}^{n}\left(\frac{i-4\sqrt{i}}{i-\sqrt{i}+3}\right)x_0 \tag{3.8}$$

Using (3.5) and (3.6), we have

$$\begin{aligned}\frac{x_n(IN)}{x_n(IM)} &= \prod_{i=25}^{n}\left(\frac{i^{3/2}-4i}{i^{3/2}-i+6\sqrt{i}-9}\right)\left(\frac{\sqrt{i}-1}{\sqrt{i}-4}\right) = \prod_{i=25}^{n}\frac{i^2-5i^{3/2}+4i}{i^2-5i^{3/2}+10i-33\sqrt{i}+36}\\ &= \prod_{i=25}^{n}\left[1 - \frac{(6i-33\sqrt{i}+36)}{i^2-5i^{3/2}+10i-33\sqrt{i}+36}\right]\end{aligned}$$

But

$$\begin{aligned}0 &\le \lim_{n\to\infty}\prod_{i=25}^{n}\left(1 - \frac{6i-33\sqrt{i}+36}{i^2-5i^{3/2}+10i-33\sqrt{i}+36}\right)\\ &\le \lim_{n\to\infty}\prod_{i=25}^{n}\left(1-\frac{1}{i}\right) = \lim_{n\to\infty}\frac{24}{25}\times\frac{25}{26}\dots\frac{n-1}{n} = \lim_{n\to\infty}\frac{24}{n} = 0.\end{aligned}$$

Hence $\lim_{n\to\infty}\left|\frac{x_n(IN)-0}{x_n(IM)-0}\right| = 0$. Therefore, using definition 1.5, the new three-step implicit iteration (IN) converges faster than the implicit Mann iteration (IM) to the fixed-point $p = 0$.

Similarly, using (3.5) and (3.7), we arrive at

$$\frac{x_n(IM)}{x_n(M)} = \prod_{i=25}^{n}\left(\frac{\sqrt{i}-4}{\sqrt{i}-1}\right)\left(\frac{\sqrt{i}}{\sqrt{i}-3}\right) = \prod_{i=25}^{n}\left(\frac{i-4\sqrt{i}}{i-4\sqrt{i}+3}\right)$$

with

$$0 \le \lim_{n\to\infty}\prod_{i=25}^{n}\left(\frac{i-4\sqrt{i}}{i-4\sqrt{i}+3}\right) \le \lim_{n\to\infty}\prod_{i=25}^{n}\left(1-\frac{1}{i}\right) = \lim_{n\to\infty}\frac{24}{25}\times\frac{25}{26}\dots\frac{n-1}{n} = \lim_{n\to\infty}\frac{24}{n} = 0.$$

Therefore $\lim_{n\to\infty}\left|\frac{x_n(IM)-0}{x_n(M)-0}\right| = 0$. That is implicit Mann iteration (IM) converges faster than the explicit Mann iteration (M) to the fixed-point $p = 0$.

Also, using (3.6) and (3.8), we get

$$\frac{x_n(IN)}{x_n(II)} = \prod_{i=25}^{n}\left(\frac{i^{3/2}-4i}{i^{3/2}-i+6\sqrt{i}-9}\right)\left(\frac{i-\sqrt{i}+3}{i-4\sqrt{i}}\right) = \prod_{i=25}^{n}\frac{i^{5/2}-i^2+3i^{3/2}-4i^2+4i^{3/2}-12i}{i^{5/2}-4i^2-i^2+4i^{3/2}+6i^{3/2}+36\sqrt{i}}$$

$$= \prod_{i=25}^{n}\frac{i^{5/2}-5i^2+7i^{3/2}-12i}{i^{5/2}-5i^2+10i^{3/2}} = \prod_{i=25}^{n}\left[1-\frac{(3i^{3/2}+12i)}{i^{5/2}-5i^2+10i^{3/2}}\right]$$

with

$$0 \le \lim_{n\to\infty}\prod_{i=25}^{n}\left[1-\frac{(3i^{3/2}+12i)}{(i^{5/2}-5i^2+10i^{3/2})}\right] \le \lim_{n\to\infty}\prod_{i=25}^{n}\left(1-\frac{1}{i}\right) \le 0$$

which implies

$$\left|\frac{x_n(IN)-0}{x_n(II)-0}\right| = 0$$

Therefore, the new three-step iteration converges fast as compared to two-step implicit Ishikawa-type iteration.

Using computer programming in C++, the convergence speed of various iterations is compared and observations are listed in the Table 1 by taking initial approximation $x_0 = 1$, $T(x) = \frac{x}{4}$ and $\alpha_n = \beta_n = \gamma_n = 1 - \frac{4}{\sqrt{n}}$, $n \ge 25$. The table reveals that newly introduced implicit iteration has better convergence rate as compared to implicit Ishikawa-type iteration, implicit Mann iteration as well as explicit Mann iteration and implicit Mann iteration converges faster than corresponding explicit Mann iteration to the fixed-point $p = 0$.

Table 1. Comparison of convergence rate of new iteration with other iterations

Number of iterations (n)	Mann iteration (M)	Implicit Mann iteration (IM)	Implicit Ishikawa type iteration (II)	Implicit new iteration (IN)
25	0.4	0.25	0.217391	0.206612
26	0.164661	0.0670294	0.0509705	0.0460629
27	0.0695938	0.0191074	0.0127723	0.0109812
28	0.0301378	0.00575025	0.0033952	0.00277867
29	0.0133485	0.00181636	0.000951573	0.000741746
30	0.00603721	0.000599294	0.000279742	0.000207812
31	0.00278426	0.000205692	8.58832e−005	6.08396e−005
32	0.00130768	7.31828e−005	2.74322e−005	1.85426e−005
33	0.000624769	2.69091e−005	9.08659e−006	5.8642e−006
34	0.000303328	1.01987e−005	3.11239e−006	1.91897e−006
35	0.000149513	3.97501e−006	1.09965e−006	6.48125e−007
36	7.47563e−005	1.59e−006	3.99872e−007	2.25435e−007
–	–	–	–	–

4. Data dependence of implicit iteration in hyperbolic spaces

THEOREM 4.1 *Let $T:K \to K$ be a mapping satisfying (1.7). Let T_1 be an approximate operator of T as in Definition 1.7, and $\{x_n\}_{n=0}^{\infty}$, $\{u_n\}_{n=0}^{\infty}$ be two implicit iterations associated to T, T_1 and defined by*

$$\begin{aligned} x_n &= W(x_{n-1}, Ty_n, \alpha_n) \\ y_n &= W(z_n, Tz_n, \beta_n) \\ z_n &= W(x_n, Tx_n, \gamma_n) \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} u_n &= W(u_{n-1}, T_1 v_n, \alpha_n) \\ v_n &= W(w_n, T_1 w_n, \beta_n) \\ w_n &= W(u_n, T_1 u_n, \gamma_n) \end{aligned} \tag{4.2}$$

respectively, where $\alpha_{nn=0}^{\infty}$, $\beta_{nn=0}^{\infty}$ *and* $\gamma_{nn=0}^{\infty}$ are real sequences in [0, 1] satisfying $\sum_{n=0}^{\infty}(1-\alpha_n) = \infty$. Let $p = Tp$ *and* $q = T_1 q$,, then for $\varepsilon > 0$, we have the following estimate:

$$d(p,q) \leq \frac{\varepsilon}{(1-a)^2}.$$

Proof Using Definition 1.2, iterations (4.1) and (4.2) yield the following estimates:

$$\begin{aligned} d(x_n, u_n) &= d(W(x_{n-1}, Ty_n, \alpha_n), W(u_{n-1}, T_1 v_n, \alpha_n)) \leq \alpha_n d(x_{n-1}, u_{n-1}) + (1-\alpha_n) d(Ty_n, T_1 v_n) \\ &\leq \alpha_n d(x_{n-1}, u_{n-1}) + (1-\alpha_n)\{d(Ty_n, T_1 y_n) + d(T_1 y_n, T_1 v_n)\} \\ &\leq \alpha_n d(x_{n-1}, u_{n-1}) + (1-\alpha_n)\{\varepsilon + \varphi d(y_n, T_1 y_n) + a d(y_n, v_n)\} \\ &\leq \alpha_n d(x_{n-1}, u_{n-1}) + (1-\alpha_n)\varepsilon + (1-\alpha_n)\varphi d(y_n, T_1 y_n) + (1-\alpha_n) a d(y_n, v_n) \end{aligned} \tag{4.3}$$

$$d(y_n, v_n) = d(W(z_n, Tz_n, \beta_n), W(w_n, T_1 w_n, \beta_n)) \leq \beta_n d(z_n, w_n) + (1-\beta_n) d(Tz_n, T_1 w_n) \tag{4.4}$$

$$d(Tz_n, T_1 w_n) \leq d(Tz_n, T_1 z_n) + d(T_1 z_n, T_1 w_n) \leq \varepsilon + \varphi d(z_n, T_1 z_n) + a d(z_n, w_n) \tag{4.5}$$

$$d(z_n, w_n) \leq \gamma_n d(x_n, u_n) + (1-\gamma_n) d(Tx_n, T_1 u_n) \tag{4.6}$$

$$d(Tx_n, T_1 u_n) \leq d(Tx_n, T_1 x_n) + d(T_1 x_n, T_1 u_n) \leq \varepsilon + \varphi d(x_n, T_1 x_n) + a d(x_n, u_n) \tag{4.7}$$

Using (4.3)–(4.7), we arrive at

$$\begin{aligned} d(x_n, u_n) &\leq \alpha_n d(x_{n-1}, u_{n-1}) \\ &+\{a(1-\alpha_n)\beta_n\gamma_n - a^2(1-\alpha_n)\beta_n(1-\gamma_n) \\ &-a^2(1-\alpha_n)(1-\beta_n)\gamma_n - a^3(1-\alpha_n)(1-\beta_n)(1-\gamma_n) d(x_n, u_n) \\ &+(1-\alpha_n)a^2\{(1-\beta_n)(1-\gamma_n)\varphi d(x_n, T_1 x_n) + \varphi d(y_n, T_1 y_n) + a(1-\beta_n)\varphi d(z_n, T_1 z_n)\} \\ &+(1-\alpha_n)\varepsilon\{a(1-\beta_n) + a^2(1-\beta_n)(1-\gamma_n) + 1\} \end{aligned}$$

which further implies

$$\begin{aligned} & d(x_n, u_n)[1 - a(1-\alpha_n)\{\beta_n\gamma_n + a\beta_n(1-\gamma_n) + a(1-\beta_n)\gamma_n + a^2(1-\beta_n)(1-\gamma_n)\}] \\ &\leq \alpha_n d(x_{n-1}, u_{n-1}) \\ &+(1-\alpha_n)\{a^2(1-\beta_n)(1-\gamma_n)\varphi d(x_n, T_1 x_n) + \varphi d(y_n, T_1 y_n) + a(1-\beta_n)\varphi d(z_n, T_1 z_n)\} \\ &+(1-\alpha_n)\varepsilon\{a(1-\beta_n) + a^2(1-\beta_n)(1-\gamma_n) + 1\} \end{aligned} \tag{4.8}$$

and so

$$\begin{aligned} &d(x_n, u_n) \\ &\leq \frac{\alpha_n}{[1-a(1-\alpha_n)\{\beta_n\gamma_n+a\beta_n(1-\gamma_n)+a(1-\beta_n)\gamma_n+a^2(1-\beta_n)(1-\gamma_n)\}]} d(x_{n-1}, u_{n-1}) \\ &\quad + \frac{(1-\alpha_n)\{a^2(1-\beta_n)(1-\gamma_n)\varphi d(x_n,T_1x_n)+\varphi d(y_n,T_1y_n)+a(1-\beta_n)\varphi d(z_n,T_1z_n\})}{[1-a(1-\alpha_n)\{\beta_n\gamma_n+a\beta_n(1-\gamma_n)+a(1-\beta_n)\gamma_n+a^2(1-\beta_n)(1-\gamma_n)\}]} \\ &\quad + \frac{(1-\alpha_n)\varepsilon\{a(1-\beta_n)+a^2(1-\beta_n)(1-\gamma_n)+1\}}{[1-a(1-\alpha_n)\{\beta_n\gamma_n+a\beta_n(1-\gamma_n)+a(1-\beta_n)\gamma_n+a^2(1-\beta_n)(1-\gamma_n)\}]}. \end{aligned} \tag{4.9}$$

Let $\frac{C_n}{D_n} = \frac{\alpha_n}{[1-a(1-\alpha_n)\{\beta_n\gamma_n+a\beta_n(1-\gamma_n)+a(1-\beta_n)\gamma_n+a^2(1-\beta_n)(1-\gamma_n)\}]}$

then

$$\begin{aligned} 1 - \frac{C_n}{D_n} &= \frac{1-a(1-\alpha_n)\{\beta_n\gamma_n+a\beta_n(1-\gamma_n)+a(1-\beta_n)\gamma_n+a^2(1-\beta_n)(1-\gamma_n)\}-\alpha_n}{[1-a(1-\alpha_n)\{\beta_n\gamma_n+a\beta_n(1-\gamma_n)+a(1-\beta_n)\gamma_n+a^2(1-\beta_n)(1-\gamma_n)\}]} \\ &\geq 1 - [a(1-\alpha_n) + \alpha_n] \end{aligned}$$

which further implies

$$\frac{C_n}{D_n} \leq a(1-\alpha_n) + \alpha_n = 1 - (1-\alpha_n)(1-a) \tag{4.10}$$

Using (4.10), (4.9) becomes

$$\begin{aligned} d(x_n, u_n) &\leq [1-(1-\alpha_n)(1-a)]d(x_{n-1}, u_{n-1}) \\ &\quad + \frac{(1-\alpha_n)(1-a)\left\{\begin{array}{l} a^2(1-\beta_n)(1-\gamma_n)\varphi d(x_n, T_1x_n) \\ +\varphi d(y_n, T_1y_n) + a(1-\beta_n)\varphi d(z_n, T_1z_n) + 3\varepsilon \end{array}\right\}}{(1-a)[1-a(1-\alpha_n)\{\beta_n\gamma_n+a\beta_n(1-\gamma_n)+a(1-\beta_n)\gamma_n+a^2(1-\beta_n)(1-\gamma_n)\}]} \end{aligned} \tag{4.11}$$

Now, it is easy to see that

$$\begin{aligned} &[1-a(1-\alpha_n)\{\beta_n\gamma_n + a\beta_n(1-\gamma_n) + a(1-\beta_n)\gamma_n + a^2(1-\beta_n)(1-\gamma_n)\}] \\ &\quad = [1-a(1-\alpha_n)\{1-(1-\beta_n)(1-a)\}\{1-(1-\gamma_n)(1-a)\}] \geq 1-a \end{aligned}$$

and hence

$$\frac{1}{[1-a(1-\alpha_n)\{1-(1-\beta_n)(1-a)\}\{1-(1-\gamma_n)(1-a)\}]} \leq \frac{1}{1-a}$$

So, (4.11) becomes

$$\begin{aligned} d(x_n, u_n) &\leq [1-(1-\alpha_n)(1-a)]d(x_{n-1}, u_{n-1}) \\ &\quad + \frac{(1-\alpha_n)(1-a)\left\{\begin{array}{l} a^2(1-\beta_n)(1-\gamma_n)\varphi d(x_n, T_1x_n) \\ +\varphi d(y_n, T_1y_n) + a(1-\beta_n)\varphi d(z_n, T_1z_n) + 3\varepsilon \end{array}\right\}}{(1-a)^2} \end{aligned} \tag{4.12}$$

or

$$a_n \leq (1-r_n)a_{n-1} + r_n t_n$$

where

$$a_n = d(x_n, u_n),\ r_n = (1-\alpha_n)(1-a)$$

and

$$t_n = \frac{\left\{\begin{array}{l} a^2(1-\beta_n)(1-\gamma_n)\varphi d(x_n, T_1x_n) \\ +\varphi d(y_n, T_1y_n) + a(1-\beta_n)\varphi d(z_n, T_1z_n) + 3\varepsilon \end{array}\right\}}{(1-a)^2}$$

Now, from Theorem 2.1, we have $\lim_{n\to\infty} d(x_n, p) = 0$, $\lim_{n\to\infty} d(u_n, p) = 0$ and since ϕ is continuous, hence $\lim_{n\to\infty} \varphi d(x_n, Tx_n) = \lim_{n\to\infty} \varphi d(y_n, Ty_n) = \lim_{n\to\infty} \varphi d(z_n, Tz_n) = 0.$

Therefore, using Lemma (1.8), (4.12) yields

$$d(p, q) \leq \frac{3\varepsilon}{(1-a)^2}$$

Remark 4.2 Putting $\gamma_n = \beta_n = 1$ and $\gamma_n = 1$ in (4.1) and (4.2), respectively, data-dependence results for implicit Mann iteration and implicit Ishikawa-type iteration can be proved easily on the same lines as in Theorem 4.1.

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