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## APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

# Initial value problem of fractional order

A. Guezane-Lakoud<sup>1\*</sup>

**Abstract:** In this work, we discuss the existence of positive solutions for a class of fractional initial value problems. For this, we rewrite the posed problem as a Volterra integral equation, then, using Guo–Krasnoselskii theorem, positivity of solutions is established under some conditions. An example is given to illustrate the obtained results.

**Subjects:** Science; Mathematics & Statistics; Applied Mathematics

**Keywords:** initial value problem; fixed point theorem; positivity of solution

**Mathematics Subject classifications:** 05C38; 15A15; 15A18; 34B10; 26A33; 34B15

### 1. Introduction

This work is devoted to the study of positive solutions for the following fractional differential equation with initial conditions

$$(P): \begin{cases} D_{0^+}^q u(t) = f(t, u(t), u'(t)), & 0 < t \leq 1, \\ u(0) = u'(0) = u''(0) = 0. \end{cases}$$

Where  $f: [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  is a given function,  $2 < q < 3$ ,  $D_{0^+}^q$  denotes the Riemann's fractional derivative. We note that few papers dealing with fractional differential equations, considered the nonlinearity  $f$  in (P) depending on the derivative of  $u$ , due to this fact we need more assumptions on  $f$  and the problem becomes more complicated.

Fractional initial value problems have been studied recently by many authors. In the paper of Yoruk, Gnana Bhaskar, and Agarwal (2013), Krasnoselskii-Krein, Nagumo's type uniqueness result and successive approximations have been extended to differential equations of fractional order  $0 < q < 1$ . Some results in literature are given for boundary value problems for ordinary differential equation, by Webb (2009) and Graef, Kong, and Wang (2008) in the case where the Green function associated to the posed problem is vanishing on a set of zero measure. By means of Guo–Krasnosel'skii fixed point theorem the existence of nontrivial positive solution is proved.

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### PUBLIC INTEREST STATEMENT

Under suitable conditions on the nonlinearity term, we prove the existence of positive solutions for an initial fractional value problem. The proofs are based on a fixed point theorem.



Existence and positivity of solutions for boundary value problems have been studied by using different methods, such as fixed point theory, topological degree methods, upper and lower solutions... (see Agarwal, O'Regan, & Staneik, 2010; Ahmad & Nieto, 2009; Cabada & Infante, 2013; Graef et al., 2008; Guezane-Lakoud & Khaldi, 2012a; 2012b; 2012c; Guo & Lakshmikantham, 1988; Henderson & Thompson, 2000; Infante & Webb, 2002; Lakshmikantham & Vatsala, 2008; Ntouyas, Wang, & Zhang, 2011; Webb, 2009; 2001; Webb & Infante, 2008).

In this work, we discuss the existence of positive solutions for the problem (P). To prove our results, we assume some conditions on the nonlinear term  $f$ , then we use a cone fixed point theorem due to Guo-Krasnoselskii.

## 2. Preliminaries

We present some definitions from fractional calculus theory which will be needed later (see Kilbas, Srivastava, & Trujillo, 2006; Podlubny, 1999).

**Definition 2.1** The Riemann-Liouville fractional integral of order  $\alpha > 0$  of a function  $g$  is defined by

$$I_{a^+}^\alpha g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g(s)}{(t-s)^{1-\alpha}} ds.$$

**Definition 2.2** The Riemann fractional derivative of order  $q$  of  $g$  is defined by

$$D_{a^+}^q g(t) = \frac{1}{\Gamma(n-q)} \left( \frac{d}{dt} \right)^n \int_a^t \frac{g(s)}{(t-s)^{q-n+1}} ds,$$

where  $n = [q] + 1$ . ( $[q]$  is the integer part of  $q$ ).

**LEMMA 2.3** The homogenous fractional differential equation  $D_{a^+}^q g(t) = 0$  has a solution

$$g(t) = c_1 t^{q-1} + c_2 t^{q-2} + \dots + c_n t^{q-n}$$

where  $c_i \in \mathbb{R}, i = 1, \dots, n$  and  $n = [q] + 1$ .

**LEMMA 2.4** Let  $p, q \geq 0, f \in L_1[a, b]$ . Then  $I_{0^+}^p I_{0^+}^q f(t) = I_{0^+}^{p+q} f(t) = I_{0^+}^q I_{0^+}^p f(t)$  (properties of semi-groups) and  $D_{a^+}^q I_{0^+}^q f(t) = f(t)$ , for all  $t \in [a, b]$ .

We start by solving an auxiliary problem which allows us to get the expression of the solution, let us consider the following linear problem ( $P_0$ ):

$$\begin{aligned} D_{0^+}^q u(t) &= y(t), \quad 0 < t \leq 1, \\ u(0) &= u'(0) = u''(0) = 0. \end{aligned} \tag{2.1}$$

**LEMMA 2.5** Assume that  $y \in C([0, 1], \mathbb{R})$ , then the problem ( $P_0$ ) has a unique solution given by:

$$u(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s) ds. \tag{2.2}$$

*Proof* Using Lemmas 2.3 and 2.4, we get :

$$u(t) = I_{0^+}^q y(t) + at^{q-1} + bt^{q-2} + ct^{q-3}. \tag{2.3}$$

The condition  $u(0) = 0$  implies that  $c = 0$ . Differentiating both sides of (2.5) and using the initial condition  $u'(0) = 0$ , it yields  $b = 0$ . The condition  $u''(0) = 0$  implies  $a = 0$ . Substituting  $a, b$  and  $c$  by their values in (2.5), we obtain

$$u(t) = I_{0^+}^q y(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} y(s) ds. \tag{2.4}$$

Let  $E$  be the Banach space of all function  $u \in C^1([0, 1])$  into  $\mathbb{R}$  with the norm  $\|u\| = \|u\|_\infty + \|u'\|_\infty$  where  $\|u\|_\infty = \max_{t \in [0, 1]} |u(t)|$ . Define the operator  $T: E \rightarrow E$  as follows:

$$Tu(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, u(s), u'(s)) ds. \tag{2.5}$$

LEMMA 2.6 *The function  $u \in E$  is solution of the initial value problem (P) if and only if  $Tu(t) = u(t)$ , for all  $t \in [0, 1]$ .*

*From here we see that to solve the FIVP (P) it remains to prove that the map  $T$  has a fixed point in  $E$ .*

### 3. Main results

First, we state the assumptions that will be used to prove the existence of positive solutions:

(H<sub>1</sub>)  $f(t, u, v) = g(t) f_1(u, v)$  where  $g \in L^1([0, 1], \mathbb{R}_+)$ ,  $f_1 \in C(\mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}_+)$ ,  $f_1(0, 0) \neq 0$ .

(H<sub>2</sub>) There exists two positive constants  $g_1$  and  $g_2$  such that  $0 < g_1 \leq g(t) \leq g_2$  for all  $t \in [0, 1]$ .

The operator  $T: E \rightarrow E$  becomes

$$Tu(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} g(s) f_1(u(s), u'(s)) ds.$$

Let us introduce the following notations

$$A_\delta = \lim_{(|u|+|v|) \rightarrow \delta} \frac{f_1(u, v)}{u+v}, (\delta = 0^+ \text{ or } +\infty).$$

Let  $K$  be the classical cone

$$K = \{u \in E, u(t) \geq 0, u'(t) \geq 0, \text{ for all } t \in [0, 1]\}.$$

Recall the definition of a positive solution:

**Definition 3.1** A function  $u$  is called positive solution of problem (P) if  $u(t) \geq 0, \forall t \in [0, 1]$  and it satisfies the differential equation and the initial conditions in (P).

Now, we give the main result of this paper

**THEOREM 3.2** *Under the assumptions (H<sub>1</sub>) and (H<sub>2</sub>) and if  $f_1$  is convex and decreasing to each variables (i.e. for  $u$  fix,  $f_1(u, \cdot)$  is decreasing according to the second variable and for  $v$  fix the function  $f_1(\cdot, v)$  is decreasing according to the first variable), then the problem (P) has at least one nontrivial positive solution in the cone  $K$ , in the case  $A_0 = +\infty$  and  $A_\infty = 0$ .*

Recall that a function  $F: \Delta = [a, b] \times [c, d] \rightarrow \mathbb{R}$  is convex on  $\Delta$  if

$$F(\lambda x + (1-\lambda)z, \lambda y + (1-\lambda)w) \leq \lambda F(x, y) + (1-\lambda)F(z, w)$$

holds for all  $(x, y), (z, w) \in \Delta$  and  $\lambda \in [0, 1]$ .

Jensen's inequality for a convex function is given by:

**THEOREM 3.3** (Zabandan & Kiliçman, 2012) Let  $p$  be a non-negative continuous function on  $[a, b]$  such that  $\int_a^b p(x)dx > 0$ . If  $g$  and  $h$  are real-valued continuous functions on  $[a, b]$  and  $m_1 \leq g(x) \leq M_1$ ,  $m_2 \leq h(x) \leq M_2$  for all  $x \in [a, b]$  and  $F$  is convex on  $\Delta = [m_1, M_1] \times [m_2, M_2]$ , then

$$F\left(\frac{\int_a^b g(t)p(t)dt}{\int_a^b p(t)dt}, \frac{\int_a^b h(t)p(t)dt}{\int_a^b p(t)dt}\right) \leq \frac{\int_a^b F(g(t), h(t))p(t)dt}{\int_a^b p(t)dt}.$$

The inequalities hold in reversed order if  $f$  is concave on  $\Delta$ .

For the proof of Theorem 3.2, we need the following results:

**LEMMA 3.4** (Wang, 2003) If  $f_1$  is continuous then  $A_0^* = A_0$  and  $A_\infty^* = A_\infty$ , where  $A^*: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ ,  $A^*(r) = \max\{f_1(u, v), 0 \leq u + v \leq r\}$  and

$$A_\delta^* = \lim_{r \rightarrow \delta} \frac{A^*(r)}{r}, \quad (\delta = 0^+ \text{ or } +\infty).$$

For the proof of Theorem 3.2, we use the following version of Guo–Krasnoselskii fixed point theorem Guo and Lakshmikantham (1988):

**THEOREM 3.5** Let  $E$  be a Banach space, and let  $K \subset E$  be a cone. Assume  $\Omega_1$  and  $\Omega_2$  are open-bounded subsets of  $E$  with  $0 \in \Omega_1$ ,  $\overline{\Omega_1} \subset \Omega_2$  and let

$$\mathcal{A}: K \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow K$$

be a completely continuous operator such that

- (i)  $\|\mathcal{A}u\| \leq \|u\|, u \in K \cap \partial\Omega_1$ , and  $\|\mathcal{A}u\| \geq \|u\|, u \in K \cap \partial\Omega_2$ ; or
- (ii)  $\|\mathcal{A}u\| \geq \|u\|, u \in K \cap \partial\Omega_1$ , and  $\|\mathcal{A}u\| \leq \|u\|, u \in K \cap \partial\Omega_2$ .

Then  $\mathcal{A}$  has a fixed point in  $K \cap (\overline{\Omega_2} \setminus \Omega_1)$ .

*Proof* of Theorem 3.2. Using Ascoli Arzela Theorem, we prove that  $T$  is a completely continuous operator. From  $A_0 = +\infty$ , we deduce that for  $M \geq \frac{\Gamma(q+2)}{g_1}$ , there exists  $r_1 > 0$ , such that if  $0 < u + v \leq r_1$  then  $f_1(u, v) \geq M(u + v)$ . Let  $\Omega_1 = \{u \in E, \|u\| < r_1\}$ , we should prove the first statement of Theorem 3.5. Assume that  $u_1 \in K \cap \partial\Omega_1$ , then the mean value theorem implies

$$\begin{aligned} \|Tu_1\| &\geq \|Tu_1\|_\infty \geq \int_0^1 Tu_1(t)dt \\ &= \frac{1}{\Gamma(q)} \int_0^1 \left( \int_0^t (t-s)^{q-1} dt \right) g(s)f_1(u_1(s), u_1'(s))ds \\ &= \frac{1}{\Gamma(q+1)} \int_0^1 (1-s)^q g(s)f_1(u_1(s), u_1'(s))ds. \end{aligned}$$

Now from the convexity of  $f_1$ , then Jensen's inequality and the assumption  $(H_2)$ , it yields

$$\begin{aligned} \|Tu_1\| &\geq \frac{1}{\Gamma(q+1)} \int_0^1 (1-s)^q g(s) f_1(u_1(s), u_1'(s)) ds \\ &\geq \frac{1}{\Gamma(q+1)} \left( \int_0^1 (1-s)^q g(s) ds \right) \\ &\quad \times f_1 \left( \frac{\int_0^1 (1-s)^q g(s) u_1(s) ds}{\int_0^1 (1-s)^q g(s) ds}, \frac{\int_0^1 (1-s)^q g(s) u_1'(s) ds}{\int_0^1 (1-s)^q g(s) ds} \right) \\ &\geq \frac{g_1}{\Gamma(q+2)} f_1 \left( \frac{\int_0^1 (1-s)^q g(s) u_1(s) ds}{\int_0^1 (1-s)^q g(s) ds}, \frac{\int_0^1 (1-s)^q g(s) u_1'(s) ds}{\int_0^1 (1-s)^q g(s) ds} \right), \end{aligned}$$

consequently

$$\|Tu_1\| \geq \frac{g_1}{\Gamma(q+2)} f_1 \left( \frac{\int_0^1 (1-s)^q g(s) u_1(s) ds}{\int_0^1 (1-s)^q g(s) ds}, \frac{\int_0^1 (1-s)^q g(s) u_1'(s) ds}{\int_0^1 (1-s)^q g(s) ds} \right). \tag{3.1}$$

Since  $\frac{\int_0^1 (1-s)^q g(s) u_1(s) ds}{\int_0^1 (1-s)^q g(s) ds} \leq \|u_1\|_\infty$ ,  $\frac{\int_0^1 (1-s)^q g(s) u_1'(s) ds}{\int_0^1 (1-s)^q g(s) ds} \leq \|u_1'\|_\infty$  and  $f_1$  is decreasing in each variables, then

(3.1) becomes

$$\begin{aligned} \|Tu_1\| &\geq \frac{g_1}{\Gamma(q+2)} f_1 \left( \frac{\int_0^1 (1-s)^q g(s) u_1(s) ds}{\int_0^1 (1-s)^q g(s) ds}, \frac{\int_0^1 (1-s)^q g(s) u_1'(s) ds}{\int_0^1 (1-s)^q g(s) ds} \right) \\ &\geq \frac{g_1}{\Gamma(q+2)} f_1 (\|u_1\|_\infty, \|u_1'\|_\infty) \geq \frac{g_1}{\Gamma(q+2)} M \|u_1\| \geq \|u_1\|. \end{aligned}$$

Secondly, taking into account Lemma 3.4 and the fact that  $A_\infty = 0$ , it results that  $A_\infty^* = 0$ , so for  $0 < \varepsilon \leq \frac{\Gamma(q+1)}{(q+1)g_2}$ , there exists  $R > 0$ , such that if  $r \geq R$  then  $A^*(r) \leq \varepsilon r$ . Let  $r_2 < \max(r_1, R)$  and set  $\Omega_2 = \{u \in E, \|u\| < r_2\}$ , it is easy to see that  $\overline{\Omega_1} \subset \Omega_2$ . Assume that  $u_2 \in K \cap \partial\Omega_2$ , then

$$\begin{aligned} \|Tu_2\| &= \frac{1}{\Gamma(q)} \max_{t \in [0,1]} \int_0^t \left( (t-s)^{q-1} + (q-1)(t-s)^{q-2} \right) g(s) f_1(u_2(s), u_2'(s)) ds \\ &\leq \frac{g_2 A^*(r_2)}{\Gamma(q)} \max_{t \in [0,1]} \int_0^t \left( (t-s)^{q-1} + (q-1)(t-s)^{q-2} \right) ds \\ &\leq \frac{g_2 (q+1) \varepsilon r_2}{\Gamma(q+1)} \leq \|u\|. \end{aligned}$$

then from the second statement of Theorem 3.5,  $T$  has a fixed point in  $K \cap (\overline{\Omega_2} \setminus \Omega_1)$ .

*Example 3.6* Let us consider the problem (P) with  $q = \frac{5}{2}$ ,  $f_1(u, v) = \frac{1}{1+u+v}$ ,  $f_1(0, 0) \neq 0$ ,  $g(t) = 1 + t^2$ ,  $g_1 = 1$ ,  $g_2 = 2$ . We check easily that  $A_0 = +\infty$  and  $A_\infty = 0$  and that the assumptions  $(H_1) - (H_2)$  are satisfied. Theorem 3.2 implies that there exists at least one nontrivial positive solution in the cone  $K$ .

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