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## PURE MATHEMATICS | RESEARCH ARTICLE

# Boundedness on Orlicz space of Toeplitz type operators related to multiplier operator and mean oscillation

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**Abstract:** In this paper, the boundedness for certain Toeplitz type operator related to the multiplier operator from Lebesgue space to Orlicz space is obtained.

**Subjects:** Science; Advanced Mathematics; Pure Mathematics

**Keywords:** Toeplitz type operator; multiplier operator; BMO space; Orlicz space

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### 1. Introduction and preliminaries

Let  $b$  be a locally integrable function on  $R^n$  and  $T$  be an integral operator. For a suitable function  $f$ , the commutator generated by  $b$  and  $T$  is defined by  $[b, T]f = bT(f) - T(bf)$ . It is well known that one important role of commutators is to characterize function spaces, which is originated by Coifman, Rochberg, and Weiss (1976). They characterized  $BMO$  space via the  $L^p$  boundedness of the commutator for singular integral operator. Since then, similar results of other operators have also been obtained (see Chanillo, 1982; Janson, 1978, Paluszynski, 1995). Now, with the development of singular integral operators (see Garcia-Cuerva & Rubio de Francia, 1985; Stein, 1993), their commutators have been well studied. In Coifman et al. (1976), Wang and Liu (2009a, 2009b), the authors proved that the commutators generated by the singular integral operators and  $BMO$  functions are bounded on  $L^p(R^n)$  for  $1 < p < \infty$ . Chanillo (1982) proved a similar result when singular integral operators are replaced by the fractional integral operators. Janson (1978) proved the boundedness for the commutators generated by the singular integral operators and  $BMO$  functions from Lebesgue spaces to Orlicz spaces. Lu and Mo (2009), some multiplier operators are introduced and the boundedness for the operators are obtained (see Kurtz & Wheeden, 1979; Muckenhoupt, Wheeden, & Young, 1987; Wang & Liu, 2009a, 2009b; You, 1988; Zhang & Chen, 2005, 2006). Krantz and Li (2001), Lu and Mo (2009), some Toeplitz type operators related to the singular integral operators are introduced, and the boundedness for the operators generated by  $BMO$  and Lipschitz functions are obtained. Motivated by these, in this paper, we will prove the boundedness properties of the Toeplitz type operator associated to the multiplier operator from Lebesgue space to Orlicz space.



Ouyang Difei

### ABOUT THE AUTHOR

Ouyang Difei has been studying the topics from 2000 year. The issue will provide some new thinking and methods for the domain. It is natural and important for the domain.

### PUBLIC INTEREST STATEMENT

It is one of the core problems in analysis mathematics of studying the boundedness of integral operators on function spaces. The Toeplitz type operators are important ones, which are the non-trivial and natural generalizations of the commutator operator. It is an advanced and hot research topics in harmonic analysis to study the boundedness of Toeplitz type operators on function spaces. In this paper, the boundedness for certain Toeplitz type operator related to the multiplier operator from Lebesgue space to Orlicz space is obtained.

First, let us introduce some notations. Throughout this paper,  $Q$  will denote a cube of  $R^n$  with sides parallel to the axes. For any locally integrable function  $f$ , the sharp function of  $f$  is defined by

$$f^\#(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy$$

where, and in what follows,  $f_Q = |Q|^{-1} \int_Q f(x) dx$ . It is well-known that (see Garcia-Cuerva & Rubio de Francia, 1985)

$$f^\#(x) \approx \sup_{Q \ni x} \inf_{c \in \mathbb{C}} \frac{1}{|Q|} \int_Q |f(y) - c| dy$$

Let  $M$  be the Hardy–Littlewood maximal operator defined by

$$M(f)(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy$$

We write that  $M_\rho f = (M(f^\rho))^{1/\rho}$  for  $0 < \rho < \infty$ . For  $1 \leq r < \infty$  and  $0 < \eta < n$ , let

$$M_{\eta,r}(f)(x) = \sup_{Q \ni x} \left( \frac{1}{|Q|^{1-\eta/n}} \int_Q |f(y)|^r dy \right)^{1/r}$$

We say that  $f$  belongs to  $BMO(R^n)$  if  $f^\#$  belongs to  $L^\infty(R^n)$  and  $\|f\|_{BMO} = \|f^\#\|_{L^\infty}$ . More generally, let  $\rho$  be a non-decreasing positive function on  $[0, +\infty)$  and define  $BMO_\rho(R^n)$  as the space of all functions  $f$  such that

$$\frac{1}{|Q(x,r)|} \int_{Q(x,r)} |f(y) - f_Q| dy \leq C\rho(r)$$

For  $\beta > 0$ , the Lipschitz space  $Lip_\beta(R^n)$  is the space of functions  $f$  such that

$$\|f\|_{Lip_\beta} = \sup_{x \neq y} |f(x) - f(y)| / |x - y|^\beta < \infty$$

For  $f$ ,  $m_f$  denotes the distribution function of  $f$ , that is  $m_f(t) = |\{x \in R^n : |f(x)| < t\}|$ .

Let  $\rho$  be a non-decreasing convex function on  $[0, +\infty)$  with  $\rho(0) = 0$ .  $\rho^{-1}$  denotes the inverse function of  $\rho$ . The Orlicz space  $L_\rho(R^n)$  is defined by the set of functions  $f$  such that  $\int_{R^n} \rho(\lambda|f(x)|) dx < \infty$  for some  $\lambda > 0$ . The norm is given by

$$\|f\|_{L_\rho} = \inf_{\lambda > 0} \lambda^{-1} \left( 1 + \int_{R^n} \rho(\lambda|f(x)|) dx \right)$$

## 2. Results

In this paper, we will study the multilinear operator as following (see Kurtz & Wheeden, 1979).

A bounded measurable function  $k$  defined on  $R^n \setminus \{0\}$  is called a multiplier. The multiplier operator  $T$  associated with  $k$  is defined by

$$T(f)(x) = k(x)\hat{f}(x), \text{ for } f \in S(R^n)$$

where  $\hat{f}$  denotes the Fourier transform of  $f$  and  $S(R^n)$  is the Schwartz test function class. Now, we recall the definition of the class  $M(s, l)$ . Denote by  $|x| \sim t$  the fact that the value of  $x$  lies in the annulus  $\{x \in R^n : ct < |x| < Ct\}$ , where  $0 < c < C < \infty$  are values specified in each instance.

**Definition 1** Let  $l \geq 0$  be a real number and  $1 \leq s \leq 2$ . we say that the multiplier  $k$  satisfies the condition  $M(s, l)$ , if

$$\left( \int_{|\xi| \sim R} |D^\alpha k(\xi)|^s d\xi \right)^{\frac{1}{s}} < CR^{n/s-l|\alpha|}$$

for all  $R > 0$  and multi-indices  $\alpha$  with  $|\alpha| \leq l$ , when  $l$  is a positive integer, and, in addition, if

$$\left( \int_{|\xi| \sim R} |D^\alpha k(\xi) - D^\alpha k(\xi - z)|^s d\xi \right)^{\frac{1}{s}} \leq C \left( \frac{|z|}{R} \right)^\gamma R^{\frac{n}{s} - |\alpha|}$$

for all  $|z| < R/2$  and all multi-indices  $\alpha$  with  $|\alpha| = [l]$ , the integer part of  $l$ , i.e.  $[l]$  is the greatest integer less than or equal to  $l$ , and  $l = [l] + \gamma$  when  $l$  is not an integer.

Denote  $D(R^n) = \{\phi \in S(R^n) : \text{supp}(\phi) \text{ is compact}\}$  and  $\hat{D}_0(R^n) = \{\phi \in S(R^n) : \hat{\phi} \in D(R^n) \text{ and } \hat{\phi} \text{ vanishes in a neighbourhood of the origin}\}$ . The following boundedness property of  $T$  on  $L^p(R^n)$  is proved by Strömberg and Torkinsky (see Kurtz & Wheeden, 1979).

**Definition 2** For a real number  $\tilde{l} \geq 0$  and  $1 \leq \tilde{s} < \infty$ , we say that  $K$  verifies the condition  $\tilde{M}(\tilde{s}, \tilde{l})$ , and write  $K \in \tilde{M}(\tilde{s}, \tilde{l})$ , if

$$\left( \int_{|x| \sim R} |D^{\tilde{\alpha}} K(x)|^{\tilde{s}} dx \right)^{\frac{1}{\tilde{s}}} \leq CR^{n/\tilde{s} - n - |\tilde{\alpha}|}, \quad R > 0$$

for all multi-indices  $|\tilde{\alpha}| \leq \tilde{l}$  and, in addition, if

$$\begin{aligned} \left( \int_{|x| \sim R} |D^{\tilde{\alpha}} K(x) - D^{\tilde{\alpha}} K(x - z)|^{\tilde{s}} dx \right)^{\frac{1}{\tilde{s}}} &\leq C \left( \frac{|z|}{R} \right)^v R^{\frac{n}{\tilde{s}} - n - u}, \quad \text{if } 0 < v < 1 \\ \left( \int_{|x| \sim R} |D^{\tilde{\alpha}} K(x) - D^{\tilde{\alpha}} K(x - z)|^{\tilde{s}} dx \right)^{\frac{1}{\tilde{s}}} &\leq C \left( \frac{|z|}{R} \right) \left( \log \frac{R}{|z|} \right) R^{\frac{n}{\tilde{s}} - n - u}, \quad \text{if } v = 1 \end{aligned}$$

for all  $|z| > \frac{R}{2}$ ,  $R < 0$ , and all multi-indices  $\tilde{\alpha}$  with  $|\tilde{\alpha}| = u$ , where  $u$  denotes the largest integer strictly less than  $\tilde{l}$  with  $\tilde{l} = u + v$ .

**LEMMA 1** (see Kurtz & Wheeden, 1979) Let  $k \in M(s, l)$ ,  $1 \leq s \leq 2$ , and  $l > \frac{n}{s}$ . Then the associated mapping  $T$ , defined a priori for  $f \in \hat{D}_0(R^n)$ ,  $T(f)(x) = (f * K)(x)$ , extends to a bounded mapping from  $L^p(R^n)$  into itself for  $1 < p < \infty$  and  $K(x) = k(x)$ .

**LEMMA 2** (see Kurtz & Wheeden, 1979) Suppose  $k \in M(s, l)$ ,  $1 \leq s \leq 2$ . Given  $1 \leq \tilde{s} < \infty$ , let  $r \geq 1$  be such that  $\frac{1}{r} = \max\{\frac{1}{s}, 1 - \frac{1}{\tilde{s}}\}$ . Then  $K \in \tilde{M}(\tilde{s}, \tilde{l})$ , where  $\tilde{l} = l - \frac{n}{r}$ .

**LEMMA 3** (see Kurtz & Wheeden, 1979) Let  $1 \leq s < \infty$ , suppose that  $\tilde{s}$  is a positive real number with  $l > n/r$ ,  $1/r = \max\{1/s, 1 - 1/\tilde{s}\}$ , and  $k \in M(s, l)$ . Then there is a positive constant  $a$ , such that

$$\left( \int_{Q_k} |K(x - z) - K(x_Q - z)|^{\tilde{s}} dz \right)^{1/\tilde{s}} \leq C 2^{-ka} (2^k h)^{-n/\tilde{s}'}$$

Now we can define the Toeplitz type operator associated to the multiplier operator as following.

**Definition 3** Let  $b$  be a locally integrable function on  $R^n$  and  $T$  be the multiplier operator. By Lemma 1,  $T(f)(x) = (K * f)(x)$  for  $K(x) = k(x)$ . The Toeplitz type operator associated to  $T$  is defined by

$$T^b = \sum_{k=1}^m T^{k,1} M^b T^{k,2}$$

where  $T^{k,1}$  are  $T$  or  $\pm I$  (the identity operator),  $T^{k,2}$  are the bounded linear operators on  $L^p(R^n)$  for  $1 < p < \infty$  and  $k = 1, \dots, m$ ,  $M^b(f) = bf$ .

Note that the commutator  $[b, T](f) = bT(f) - T(bf)$  is a particular operator of the Toeplitz type operator  $T^b$ . It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see Janson, 1978; Janson & Peetre, 1988; Krantz & Li, 2001; Paluszynski, 1995; Pérez & Pradolini, 2001; Pérez & Trujillo-Gonzalez, 2002). The main purpose of this paper is to prove the boundedness properties for the Toeplitz type operator  $T^b$  from Lebesgue spaces to Orlicz spaces.

We shall prove the following results in Section 4.

**THEOREM 1** Let  $T$  be the multiplier operator as Definition 3. Suppose that  $Q = Q(x_0, d)$  is a cube with  $\text{supp } f \subset (2Q)^c$  and  $x, \tilde{x} \in Q$ .

(1) If  $b \in BMO(\mathbb{R}^n)$ , then

$$|T^{(b-b_0)\chi_{(2Q)^c}}(f)(x) - T^{(b-b_0)\chi_{(2Q)^c}}(f)(x_0)| \leq C \|b\|_{BMO} \sum_{k=1}^m M_r(T^{k,2}(f))(\tilde{x}) \text{ for any } r > 1;$$

(II)(2) If  $0 < \beta \leq 1$  and  $b \in Lip_\beta(\mathbb{R}^n)$ , then

$$|T^{(b-b_0)\chi_{(2Q)^c}}(f)(x) - T^{(b-b_0)\chi_{(2Q)^c}}(f)(x_0)| \leq C \|b\|_{Lip_\beta} \sum_{k=1}^m M_{\beta,r}(T^{k,2}(f))(\tilde{x}) \text{ for any } r > 1.$$

**THEOREM 2** Let  $0 < \beta \leq 1, 1 < p < n/\beta$  and  $\varphi, \psi$  be two non-decreasing positive functions on  $[0, +\infty)$  with  $(\psi^1)^{-1}(t) = t^{1/p} \varphi^1(t^{-1/n})$ . Suppose that  $\psi$  is convex,  $\psi(0) = 0, \psi(2t) \leq C\psi(t)$ . Let  $T$  be the multiplier operator as Definition 3. If  $T^1(g) = 0$  for any  $g \in L^u(\mathbb{R}^n) (1 < u < \infty)$ , then  $T^b$  is bounded from  $L^p(\mathbb{R}^n)$  to  $L_{\psi^1}(\mathbb{R}^n)$  if  $b \in BMO(\mathbb{R}^n)$ .

**COROLLARY 1** Let  $0 < \beta \leq 1, 1 < p < n/\beta$  and  $T$  be the multiplier operator as Definition 3. If  $T^1(g) = 0$  for any  $g \in L^u(\mathbb{R}^n) (1 < u < \infty)$ , then  $T^b$  is bounded on  $L^p(\mathbb{R}^n)$  if  $b \in BMO(\mathbb{R}^n)$ .

**COROLLARY 2** Let  $1 < p < s < \infty$  and  $T$  be the multiplier operator as Definition 3. If  $T^1(g) = 0$  for any  $g \in L^u(\mathbb{R}^n) (1 < u < \infty)$ , then  $T^b$  is bounded from  $L^p(\mathbb{R}^n)$  to  $L^s(\mathbb{R}^n)$  if  $b \in BMO(\mathbb{R}^n)$ .

### 3. Some lemma

We need the following preliminary lemmas.

**LEMMA 4** (see Kurtz & Wheeden, 1979) Let  $T$  be the multiplier operator as Definition 3. Then  $T$  is bounded on  $L^p(\mathbb{R}^n)$  for  $1 < p < \infty$ .

**LEMMA 5** (see Garcia-Cuerva & Rubio de Francia, 1985) Let  $0 < p < \infty$ . Then, for any smooth function  $f$  for which the left-hand side is finite,

$$\int_{\mathbb{R}^n} M(f)(x)^p dx \leq C \int_{\mathbb{R}^n} f^\#(x)^p dx$$

**LEMMA 6** (see Chanillo, 1982) Suppose that  $0 < \eta < n, 1 \leq r < p < n/\eta$  and  $1/s = 1/p - \eta/n$ . Then  $\|M_{\eta,r}(f)\|_{L^s} \leq C \|f\|_{L^p}$ .

**LEMMA 7** (see Janson, 1978) Let  $\rho$  be a non-decreasing positive function on  $[0, +\infty)$  and  $\eta$  be an infinitely differentiable function on  $\mathbb{R}^n$  with compact support such that  $\int_{\mathbb{R}^n} \eta(x) dx = 1$ . Denote that  $b_t(x) = \int_{\mathbb{R}^n} b(x - ty)\eta(y) dy$ . Then  $\|b - b_t\|_{BMO} \leq C\rho(t) \|b\|_{BMO}$ .

LEMMA 8 (see Janson, 1978) Let  $0 < \beta < 1$  or  $\beta = 1$  and  $\rho$  be a non-decreasing positive function on  $[0, +\infty)$ . Then  $\|b_t\|_{Lip_\rho} \leq Ct^{-\beta} \rho(t) \|b\|_{BMO_\rho}$ :

LEMMA 9 (see Janson, 1978) Suppose  $1 \leq p_2 < p < p_1 < \infty$ ,  $\rho$  is a non-increasing function on  $R^+$ ,  $B$  is a linear operator such that  $m_{B(f)}(t^{1/p_1} \rho(t)) \leq Ct^{-1}$  if  $\|f\|_{L^{p_1}} \leq 1$  and  $m_{B(f)}(t^{1/p_2} \rho(t)) \leq Ct^{-1}$  if  $\|f\|_{L^{p_2}} \leq 1$ . Then  $\int_0^\infty m_{B(f)}(t^{1/p} \rho(t)) dt \leq C$  if  $\|f\|_{L^p} \leq (p/p_1)^{1/p}$ .

#### 4. Proofs of theorems

Now we are in position to prove our results.

*Proof of Theorem 1* For  $\text{supp} f \subset (2Q)^c$  and  $x, \tilde{x} \in Q$ , note that  $|x - y| \sim |x_0 - y|$  for  $x \in Q$  and  $y \in R^n \setminus 2Q$ . We have

$$\begin{aligned} & |T^{(b-b_Q)X_{(2Q)^c}}(f)(x) - T^{(b-b_Q)X_{(2Q)^c}}(f)(x_0)| \\ & \leq \sum_{k=1}^m |T^{k,1} M^{(b-b_Q)X_{(2Q)^c}} T^{k,2}(f)(x) - T^{k,1} M^{(b-b_Q)X_{(2Q)^c}} T^{k,2}(f)(x_0)| \end{aligned}$$

(I) By the Hölder's inequality and Lemma 3, we obtain, for  $1 < s, t < \infty$  with  $1/r + 1/s + 1/t = 1$ ,

$$\begin{aligned} & |T^{k,1} M^{(b-b_Q)X_{(2Q)^c}} T^{k,2}(f)(x) - T^{k,1} M^{(b-b_Q)X_{(2Q)^c}} T^{k,2}(f)(x_0)| \\ & \leq \int_{(2Q)^c} |b(y) - b_Q| |K(x - y) - K(x_0 - y)| |T^{k,2}(f)(y)| dy \\ & \leq \sum_{j=1}^\infty \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |b(y) - b_Q| |K(x - y) - K(x_0 - y)| |T^{k,2}(f)(y)| dy \\ & \leq C \sum_{j=1}^\infty \left( \int_{2^{j+1}Q} |b(y) - b_Q|^s dy \right)^{1/s} \left( \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^r dy \right)^{1/r} \\ & \quad \times \left( \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |K(x - y) - K(x_0 - y)|^t dy \right)^{1/t} dx \\ & \leq C \|b\|_{BMO} \sum_{j=1}^\infty j 2^{-aj} \left( \frac{1}{|2^{j+1}Q|} \int_{2^{j+1}Q} |T^{k,2}(f)(y)|^r dy \right)^{1/r} \\ & \leq C \|b\|_{BMO} M_r(T^{k,2}(f))(\tilde{x}) \end{aligned}$$

thus

$$\begin{aligned} & |T^{(b-b_Q)X_{(2Q)^c}}(f)(x) - T^{(b-b_Q)X_{(2Q)^c}}(f)(x_0)| \\ & \leq \sum_{k=1}^m |T^{k,1} M^{(b-b_Q)X_{(2Q)^c}} T^{k,2}(f)(x) - T^{k,1} M^{(b-b_Q)X_{(2Q)^c}} T^{k,2}(f)(x_0)| \\ & \leq C \|b\|_{BMO} \sum_{k=1}^m M_r(T^{k,2}(f))(\tilde{x}) \end{aligned}$$

(II) Note that, for  $b \in Lip_\beta(R^n)$ ,

$$|b(x) - b_Q| \leq \frac{1}{|Q|} \int_Q \|b\|_{Lip_\beta} |x - y|^\beta dy \leq C \|b\|_{Lip_\beta} (|x - x_0| + d)^\beta$$

similar to the proof of (I), we obtain, for  $1/r + 1/r' = 1$ ,

$$\begin{aligned} & |T^{k,1} M^{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x) - T^{k,1} M^{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x_0)| \\ & \leq \sum_{j=1}^{\infty} \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |b(y) - b_Q| |K(x-y) - K(x_0-y)| |T^{k,2}(f)(y)| dy \\ & \leq C \|b\|_{Lip_\beta} \sum_{j=1}^{\infty} |2^{j+1} Q|^{\beta/n} \left( \int_{2^j d \leq |y-x_0| < 2^{j+1} d} |K(x,y) - K(x_0,y)|^r dy \right)^{1/r'} \\ & \quad \times \left( \int_{2^{j+1} Q} |T^{k,2}(f)(y)|^r dy \right)^{1/r} \\ & \leq C \|b\|_{Lip_\beta} \sum_{j=1}^{\infty} 2^{-j} \left( \frac{1}{|2^{j+1} Q|^{1-r\beta/n}} \int_{2^{j+1} Q} |T^{k,2}(f)(y)|^r dy \right)^{1/r} \\ & \leq C \|b\|_{Lip_\beta} M_{\beta,r}(T^{k,2}(f))(\tilde{x}) \end{aligned}$$

thus

$$\begin{aligned} & |T^{(b-b_Q)\chi_{(2Q)^c}}(f)(x) - T^{(b-b_Q)\chi_{(2Q)^c}}(f)(x_0)| \\ & \leq \sum_{k=1}^m |T^{k,1} M^{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x) - T^{k,1} M^{(b-b_Q)\chi_{(2Q)^c}} T^{k,2}(f)(x_0)| \\ & \leq C \|b\|_{Lip_\beta} \sum_{k=1}^m M_{\beta,r}(T^{k,2}(f))(\tilde{x}) \end{aligned}$$

These complete the proof.

*Proof of Theorem 2* Without loss of generality, we may assume  $T^{k,1}$  are  $T(k=1, \dots, m)$ . We prove the theorem in several steps. First, we prove, if  $b \in BMO(R^n)$ ,

$$(T^b(f))^\# \leq C \|b\|_{BMO} \sum_{k=1}^m M_r(T^{k,2}(f)) \tag{1}$$

for any  $1 < r < \infty$ .

Fix a cube  $Q = Q(x_0, d)$  and  $\tilde{x} \in Q$ . By  $T^1(g) = 0$ , we have  $T^b(f) = T^{b-b_Q}(f)$ , thus

$$T^b(f) = T^{b-b_Q}(f) = T^{(b-b_Q)\chi_{2Q}}(f) + T^{(b-b_Q)\chi_{(2Q)^c}}(f) =: I_1(x) + I_2(x)$$

and

$$\frac{1}{|Q|} \int_Q |T^b(f)(x) - I_2(x_0)| dx \leq \frac{1}{|Q|} \int_Q |I_1(x)| dx + \frac{1}{|Q|} \int_Q |I_2(x) - I_2(x_0)| dx = I_1 + I_2$$

For  $I_1$ , choose  $1 < s < r$ , by Hölder's inequality and the boundedness of  $T$  (see Lemma 4), we obtain

$$\begin{aligned}
 & \frac{1}{|Q|} \int_Q |T^{k,1} M^{(b-b_Q)_{X_{2Q}}} T^{k,2}(f)(x)| dx \\
 & \leq \left( \frac{1}{|Q|} \int_{R^n} |T^{k,1} M^{(b-b_Q)_{X_{2Q}}} T^{k,2}(f)(x)|^s dx \right)^{1/s} \\
 & \leq C|Q|^{-1/s} \left( \int_{R^n} |M^{(b-b_Q)_{X_{2Q}}} T^{k,2}(f)(x)|^s dx \right)^{1/s} \\
 & \leq C|Q|^{-1/s} \left( \int_{2Q} |T^{k,2}(f)(x)|^r dx \right)^{1/r} \left( \int_{2Q} |b(x)-b_Q|^{rs/(r-s)} dx \right)^{(r-s)/rs} \\
 & \leq C\|b\|_{BMO} \left( \frac{1}{|Q|} \int_{2Q} |T^{k,2}(f)(x)|^r dx \right)^{1/r} \\
 & \leq C\|b\|_{BMO} M_r(T^{k,2}(f))(\bar{x})
 \end{aligned}$$

thus

$$\begin{aligned}
 I_1 & \leq \sum_{k=1}^m \frac{1}{|Q|} \int_Q |T^{k,1} M^{(b-b_Q)_{X_{2Q}}} T^{k,2}(f)(x)| dx \\
 & \leq C\|b\|_{BMO} \sum_{k=1}^m M_r(T^{k,2}(f))(\bar{x})
 \end{aligned}$$

For  $I_2$ , by using Theorem 1,

$$I_2 \leq C\|b\|_{BMO} \sum_{k=1}^m M_r(T^{k,2}(f))(\bar{x})$$

We now put these estimates together and take the supremum over all  $Q$  such that  $\bar{x} \in Q$ , we obtain

$$(T^b(f))^\#(\bar{x}) \leq C\|b\|_{BMO} \sum_{k=1}^m M_r(T^{k,2}(f))(\bar{x})$$

Thus, taking  $r$  such that  $1 < r < p$ , we obtain, by Lemma 5,

$$\begin{aligned}
 \|T^b(f)\|_{L^p} & \leq \|M(T^b(f))\|_{L^p} \leq C\|(T^b(f))^\#\|_{L^p} \\
 & \leq C\|b\|_{BMO} \sum_{k=1}^m \|M_r(T^{k,2}(f))\|_{L^p} \\
 & \leq C\|b\|_{BMO} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^p} \\
 & \leq C\|b\|_{BMO} \|f\|_{L^p}
 \end{aligned} \tag{2}$$

Secondly, we prove that, if  $b \in Lip_\beta(R^n)$ ,

$$(T^b(f))^\# \leq C\|b\|_{Lip_\beta} \sum_{k=1}^m M_{\beta,r}(T^{k,2}(f)) \tag{3}$$

for any  $r$  with  $1 < r < n/\beta$ . In fact, similar to the proof of (1) and by Theorem 1, we obtain

$$\begin{aligned}
 & \frac{1}{|Q|} \int_Q |T^b(f)(x) - I_2(x_0)| dx \\
 & \leq \sum_{k=1}^m \left( \frac{1}{|Q|} \int_{R^n} |T^{k,1} M^{(b-b_Q)_{X_{2Q}}} T^{k,2}(f)(x)|^r dx \right)^{1/r} \\
 & \quad + \frac{1}{|Q|} \int_Q |I_2(x) - I_2(x_0)| dx \\
 & \leq C \sum_{k=1}^m |Q|^{-1/r} \left( \int_{2Q} (|b(x) - b_Q| |T^{k,2}(f)(x)|)^r dx \right)^{1/r} \\
 & \quad + \sum_{k=1}^m \frac{1}{|Q|} \int_Q |T^{k,1} M^{(b-b_Q)_{X_{2Q}^c}} T^{k,2}(f)(x) - T^{k,1} M^{(b-b_Q)_{X_{2Q}^c}} T^{k,2}(f)(x_0)| dx \\
 & \leq C \sum_{k=1}^m |Q|^{-1/r} \|b\|_{Lip_\beta} |2Q|^{\beta/n} |Q|^{1/r-\beta/n} \left( \frac{1}{|Q|^{1-r\beta/n}} \int_{2Q} |T^{k,2}(f)(x)|^r dx \right)^{1/r} \\
 & \quad + C \|b\|_{Lip_\beta} \sum_{k=1}^m M_{\beta,r}(T^{k,2}(f))(\tilde{x}) \\
 & \leq C \|b\|_{Lip_\beta} \sum_{k=1}^m M_{\beta,r}(T^{k,2}(f))(\tilde{x})
 \end{aligned}$$

Thus, (3) holds. We take  $1 < r < p < n/\beta$ ,  $1/q = 1/p - \beta/n$  and obtain, by Lemma 6,

$$\begin{aligned}
 \|T^b(f)\|_{L^q} & \leq \|M(T^b(f))\|_{L^q} \leq C \|(T^b(f))^\#\|_{L^q} \\
 & \leq C \|b\|_{Lip_\beta} \sum_{k=1}^m \|M_{\beta,r}(T^{k,2}(f))\|_{L^q} \\
 & \leq C \|b\|_{Lip_\beta} \sum_{k=1}^m \|T^{k,2}(f)\|_{L^p} \\
 & \leq C \|b\|_{Lip_\beta} \|f\|_{L^p}
 \end{aligned} \tag{4}$$

Now we verify that  $T^b$  satisfies the conditions of Lemma 9. In fact, for any  $1 < p_i < n/\beta$ ,  $1/q_i = 1/p_i - \beta/n$  ( $i=1,2$ ) and  $\|f\|_{L^{p_i}} \leq 1$ , note that  $T^b(f)(x) = T^{b-b_s}(f)(x) + T^{b_s}(f)(x)$ ,  $b - b_s \in BMO(R^n)$  and  $b_s \in Lip_\beta(R^n)$ , by (2) and Lemma 7, we obtain

$$\begin{aligned}
 \|T^{b-b_s}(f)\|_{L^{p_i}} & \leq C \|b - b_s\|_{BMO} \|f\|_{L^{p_i}} \\
 & \leq C \|b - b_s\|_{BMO} \leq C \|b\|_{BMO_\varphi} \varphi(s)
 \end{aligned}$$

and by (4) and Lemma 8, we obtain

$$\|T^{b_s}(f)\|_{L^{q_i}} \leq C \|b\|_{Lip_\beta} \|f\|_{L^{p_i}} \leq C s^{-\beta} \varphi(s) \|b\|_{BMO_\varphi}$$

Thus, for  $s = t^{-1/n}$  and  $i = 1, 2$ ,

$$\begin{aligned}
 m_{T^b(f)}(\psi^{-1}(t)) & \leq m_{T^b(f)}(t^{1/p_i} \varphi(t^{-1/n})) \\
 & \leq m_{T^{b-b_s}(f)}(t^{1/p_i} \varphi(t^{-1/n})/2) + m_{T^{b_s}(f)}(t^{1/p_i} \varphi(t^{-1/n})/2) \\
 & \leq C \left[ \left( \frac{\varphi(s)}{t^{1/p_i} \varphi(s)} \right)^{p_i} + \left( \frac{s^{-\beta} \varphi(s)}{t^{1/p_i} \varphi(s)} \right)^{q_i} \right] = Ct^{-1}
 \end{aligned}$$

Taking  $1 < p_2 < p < p_1 < n/\beta$  and by Lemma 9, we obtain, for  $\|f\|_{L^p} \leq (p/p_1)^{1/p}$ ,



$$\int_{R^n} \psi(|T^b(f)(x)|) dx = \int_0^\infty m_{T^b(f)}(\psi^{-1}(t)) dt \leq C$$

then,  $\|T^b(f)\|_{L_\psi} \leq C$ .

This completes the proof of the theorem.

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