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PURE MATHEMATICS | RESEARCH ARTICLE

Sharp bounds for the Neuman-Sándor mean in terms of the power and contraharmonic means

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Abstract: In the paper, the authors obtain sharp bounds for the Neuman-Sándor mean in terms of the power and contraharmonic means.

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1. Introduction

For positive numbers $a, b > 0$ with $a \neq b$, the second Seiffert mean $T(a, b)$, quadratic mean $S(a, b)$, Neuman-Sándor mean $M(a, b)$, and contraharmonic mean $C(a, b)$ are respectively defined in Neuman and Sándor (2003), and Seiffert (1995) by

$$T(a, b) = \frac{a - b}{2 \arctan[(a - b)/(a + b)]}, \quad S(a, b) = \sqrt{\frac{a^2 + b^2}{2}} \quad (1.1)$$

$$M(a, b) = \frac{a - b}{2 \operatorname{arcsinh}[(a - b)/(a + b)]}, \quad C(a, b) = \frac{a^2 + b^2}{a + b} \quad (1.2)$$

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PUBLIC INTEREST STATEMENT

In the paper, the authors obtain a sharp lower bound and a sharp upper bound for the Neuman-Sándor mean in terms of the power and contraharmonic means.



It is well known Neuman (2012, 2011), and Neuman and Sándor (2006) that the inequalities $M(a, b) < T(a, b) < S(a, b) < C(a, b)$ hold for all $a, b < 0$ with $a \neq b$.

In Chu and Hou (2012), Chu, Hou, and Shen (2012), the inequalities

$$S(\alpha a + (1 - \alpha)b, \alpha b + (1 - \alpha)a) < T(a, b) < S(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a) \tag{1.3}$$

and

$$C(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a) < T(a, b) < C(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a) \tag{1.4}$$

were proved to be valid for $\frac{1}{2} < \alpha, \beta, \lambda, \mu < 1$ and for all $a, b < 0$ with $a \neq b$ if and only if

$$\alpha \leq \frac{1}{2} \left(1 + \sqrt{\frac{16}{\pi^2} - 1} \right), \quad \beta \geq \frac{3 + \sqrt{6}}{6} \tag{1.5}$$

$$\lambda \leq \frac{1}{2} \left(1 + \sqrt{\frac{4}{\pi} - 1} \right), \quad \mu \geq \frac{3 + \sqrt{3}}{6}$$

respectively. In Jiang and Qi (2014) and its preprint Jiang and Qi (2013a), the double inequality

$$S(\alpha a + (1 - \alpha)b, \alpha b + (1 - \alpha)a) < M(a, b) < S(\beta a + (1 - \beta)b, \beta b + (1 - \beta)a) \tag{1.6}$$

was proved to be valid for $\frac{1}{2} < \alpha, \beta < 1$ and for all $a, b > 0$ with $a \neq b$ if and only if

$$\alpha \leq \frac{1}{2} \left\{ 1 + \sqrt{\frac{1}{[\ln(1 + \sqrt{2})]^2} - 1} \right\} \quad \text{and} \quad \beta \geq \frac{3 + \sqrt{3}}{6} \tag{1.7}$$

For more information on this topic, please refer to recently published papers Chu, Wang, and Gong (2011), Jiang and Qi (2012), Li, Long, and Chu (2012), Li and Qi (2013) and references cited therein.

For $t \in (\frac{1}{2}, 1)$ and $p \geq \frac{1}{2}$, let

$$Q_{t,p}(a, b) = C^p(ta + (1 - t)b, tb + (1 - t)a)A^{1-p}(a, b) \tag{1.8}$$

where $A(a, b) = \frac{a+b}{2}$ is the classical arithmetic mean of a and b . Then, by definitions in (1.1) and (1.2), it is easy to see that

$$Q_{t,1/2}(a, b) = S(ta + (1 - t)b, tb + (1 - t)a)$$

$$Q_{t,1}(a, b) = C(ta + (1 - t)b, tb + (1 - t)a)$$

and $Q_{t,p}(a, b)$ is strictly increasing with respect to $t \in (\frac{1}{2}, 1)$.

Motivating by results mentioned above, we naturally ask a question: what are the greatest value $t_1 = t_1(p)$ and the least value $t_2 = t_2(p)$ in $(\frac{1}{2}, 1)$ such that the double inequality

$$Q_{t_1,p}(a, b) < M(a, b) < Q_{t_2,p}(a, b) \tag{1.9}$$

holds for all $a, b < 0$ with $a \neq b$ and for all $p \geq \frac{1}{2}$?

The aim of this paper is to answer this question. The solution to this question may be stated as the following Theorem 1.1.

THEOREM 1.1 Let $t_1, t_2 \in (\frac{1}{2}, 1)$ and $p \in [\frac{1}{2}, \infty)$. Then the double inequality (1.9) holds for all $a, b < 0$ with $a \neq b$ if and only if

$$t_1 \leq \frac{1}{2} \left[1 + \sqrt{\left(\frac{1}{t^*}\right)^{1/p} - 1} \right] \quad \text{and} \quad t_2 \geq \frac{1}{2} \left(1 + \frac{1}{\sqrt{6p}} \right) \quad (1.10)$$

where

$$t^* = \ln(1 + \sqrt{2}) = 0.88 \dots \quad (1.11)$$

Remark 1.1 When $p = \frac{1}{2}$ in Theorem 1.1, the double inequality (1.9) becomes (1.6).

Remark 1.2 If taking $p = 1$ in Theorem 1.1, we can conclude that the double inequality

$$C(\lambda a + (1 - \lambda)b, \lambda b + (1 - \lambda)a) < M(a, b) < C(\mu a + (1 - \mu)b, \mu b + (1 - \mu)a) \quad (1.12)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if

$$\frac{1}{2} < \lambda \leq \frac{1}{2} \left[1 + \sqrt{\frac{1}{\ln(1 + \sqrt{2})} - 1} \right] \quad \text{and} \quad 1 > \mu \geq \frac{1}{2} \left(1 + \frac{\sqrt{6}}{6} \right) \quad (1.13)$$

Remark 1.3 We note that the paper Li and Zheng (2013) is worth to being read.

2. Lemmas

In order to prove Theorem 1.1, we need the following lemmas.

LEMMA 2.1 (Anderson, Vamanamurthy, & Vuorinen, 1997, Theorem 1.25) For $-\infty < a < b < \infty$, let $f, g: [a, b] \rightarrow \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) . If $g'(x) \neq 0$ and $\frac{f'(x)}{g'(x)}$ is strictly increasing (or strictly decreasing, respectively) on (a, b) , so are the functions

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)} \quad (2.1)$$

LEMMA 2.2 The function

$$h(x) = \frac{(1 + x^2) \operatorname{arcsinh} x}{x} \quad (2.2)$$

is strictly increasing and convex on $(0, \infty)$.

Proof This follows from the following arguments:

$$h'(x) = \frac{x\sqrt{1+x^2} - \operatorname{arcsinh} x + x^2 \operatorname{arcsinh} x}{x^2} \triangleq \frac{h_1(x)}{x^2}$$

$$h'_1(x) = x \left(\frac{3x}{\sqrt{1+x^2}} + 2 \operatorname{arcsinh} x \right) \triangleq x h_2(x)$$

$$h'_2(x) = \frac{5 + 2x^2}{(1+x^2)^{3/2}} < 0$$

on $(0, \infty)$ and

$$\lim_{x \rightarrow 0^+} h_1(x) = \lim_{x \rightarrow 0^+} h_2(x) = 0$$

LEMMA 2.3 For $u \in [0, 1]$ and $p \geq \frac{1}{2}$, let

$$f_{u,p}(x) = p \ln(1 + ux^2) - \ln x + \ln \operatorname{arcsinh} x \tag{2.3}$$

on $(0, 1)$. Then the function $f_{u,p}(x)$ is positive if and only if $6pu \geq 1$ and it is negative if and only if $1 + u \leq \left(\frac{1}{t^*}\right)^{1/p}$, where t^* is defined by (1.11).

Proof It is ready that

$$\lim_{x \rightarrow 0^+} f_{u,p}(x) = 0 \tag{2.4}$$

and

$$\lim_{x \rightarrow 1^-} f_{u,p}(x) = p \ln(1 + u) + \ln(t^*) \tag{2.5}$$

An easy computation yields

$$\begin{aligned} f'_{u,p}(x) &= \frac{2pux}{1+ux^2} + \frac{1}{\sqrt{1+x^2} \operatorname{arcsinh} x} - \frac{1}{x} \\ &= \frac{u[(2p-1)x^2\sqrt{1+x^2} \operatorname{arcsinh} x + x^3] - [\sqrt{1+x^2} \operatorname{arcsinh} x - x]}{x(1+ux^2)\sqrt{1+x^2} \operatorname{arcsinh} x} \\ &= \frac{(2p-1)x^2\sqrt{1+x^2} \operatorname{arcsinh} x + x^3}{x(1+ux^2)\sqrt{1+x^2} \operatorname{arcsinh} x} \left[u - \frac{g_1(x)}{g_2(x)} \right] \end{aligned} \tag{2.6}$$

where

$$g_1(x) = \operatorname{arcsinh} x - \frac{x}{\sqrt{1+x^2}} \quad \text{and} \quad g_2(x) = (2p-1)x^2 \operatorname{arcsinh} x + \frac{x^3}{\sqrt{1+x^2}}$$

Furthermore, we have

$$g_1(0) = g_2(0) = 0 \tag{2.7}$$

and

$$\frac{g'_1(x)}{g'_2(x)} = \frac{1}{2(2p-1)\sqrt{1+x^2} h(x) + (2p+1)x^2 + 2p+2} \tag{2.8}$$

where $h(x)$ is defined by (2.2). From Lemma 2.2, it follows that the quotient $\frac{g'_1(x)}{g'_2(x)}$ is strictly decreasing on $(0, 1)$. Accordingly, from Lemma 2.1 and (2.7), it is deduced that the ratio $\frac{g_1(x)}{g_2(x)}$ is strictly decreasing on $(0, 1)$.

Moreover, making use of L'Hôpital's rule leads to

$$\lim_{x \rightarrow 0} \frac{g_1(x)}{g_2(x)} = \frac{1}{6p} \tag{2.9}$$

and

$$\lim_{x \rightarrow 1} \frac{g_1(x)}{g_2(x)} = \frac{\sqrt{2}t^* - 1}{\sqrt{2}(2p-1)t^* + 1} \tag{2.10}$$

When $u \geq \frac{1}{6p}$, combining (2.6) and (2.9) with the monotonicity of $\frac{g_1(x)}{g_2(x)}$ shows that the function $f_{u,p}(x)$ is strictly increasing on $(0, 1)$. Therefore, the positivity of $f_{u,p}(x)$ on $(0, 1)$ follows from (2.4) and the increasingly monotonicity of $f_{u,p}(x)$.

When $u \leq \frac{\sqrt{2}t^* - 1}{\sqrt{2}(2p-1)t^* + 1}$, combining (2.6) and (2.10) with the monotonicity of $\frac{g_1(x)}{g_2(x)}$ reveals that the function $f_{u,p}(x)$ is strictly decreasing on $(0, 1)$. Hence, the negativity of $f_{u,p}(x)$ on $(0, 1)$ follows from (2.4) and the decreasingly monotonicity of $f_{u,p}(x)$.

When $\frac{\sqrt{2}t^* - 1}{\sqrt{2}(2p-1)t^* + 1} < u < \frac{1}{6p}$, from (2.6), (2.9), (2.10) and the monotonicity of the ratio $\frac{g_1(x)}{g_2(x)}$, we conclude that there exists a number $x_0 \in (0, 1)$ such that $f_{u,p}(x)$ is strictly decreasing in $(0, x_0)$ and strictly increasing in $(x_0, 1)$. Denote the limit in (2.5) by $h_p(u)$. Then, from the above arguments, it follows that

$$h_p\left(\frac{1}{6p}\right) = p \ln\left(1 + \frac{1}{6p}\right) + \ln(t^*) > 0 \tag{2.11}$$

and

$$h_p\left(\frac{\sqrt{2}t^* - 1}{\sqrt{2}(2p-1)t^* + 1}\right) = p \ln\left[1 + \frac{\sqrt{2}t^* - 1}{\sqrt{2}(2p-1)t^* + 1}\right] + \ln(t^*) < 0 \tag{2.12}$$

Since $h_p(u)$ is strictly increasing for $u > -1$, so it is also in $\left[\frac{\sqrt{2}t^* - 1}{\sqrt{2}(2p-1)t^* + 1}, \frac{1}{6p}\right]$. Thus, the inequalities in (2.11) and (2.12) imply that the function $h_p(u)$ has a unique zero point $u_0 = \left(\frac{1}{t^*}\right)^{1/p} - 1 \in \left(\frac{\sqrt{2}t^* - 1}{\sqrt{2}(2p-1)t^* + 1}, \frac{1}{6p}\right)$ such that $h_p(u) < 0$ for $u \in \left[\frac{\sqrt{2}t^* - 1}{\sqrt{2}(2p-1)t^* + 1}, u_0\right)$ and $h_p(u) > 0$ for $u \in (u_0, \frac{1}{6p}]$. As a result, combining (2.4) and (2.5) with the piecewise monotonicity of $f_{u,p}(x)$ reveals that $f_{u,p}(x) < 0$ for all $x \in (0, 1)$ if and only if $\frac{\sqrt{2}t^* - 1}{\sqrt{2}(2p-1)t^* + 1} < u < u_0$. The proof of Lemma 2.3 is complete.

3. Proof of Theorem 1.1

Now we are in a position to prove our Theorem 1.1.

Since both $Q_{t,p}(a, b)$ and $M(a, b)$ are symmetric and homogeneous of degree 1, without loss of generality, we assume that $a > b$. Let $x = \frac{a-b}{a+b} \in (0, 1)$. From (1.2) and (1.8), we obtain

$$\begin{aligned} \ln \frac{Q_{t,p}(a, b)}{T(a, b)} &= \ln \frac{Q_{t,p}(a, b)}{A(a, b)} - \ln \frac{T(a, b)}{A(a, b)} \\ &= p \ln[1 + (1 - 2t)^2 x^2] - \ln x + \ln \operatorname{arcsinh} x \end{aligned}$$

Thus, Theorem 1.1 follows from Lemma 2.3.

Remark 3.1 This is a slightly modified version of the preprint Jiang and Qi (2013b).

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