



## APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

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## APPLIED & INTERDISCIPLINARY MATHEMATICS | RESEARCH ARTICLE

# A class of completely monotonic functions involving the gamma and polygamma functions

Bai-Ni Guo<sup>1,\*</sup> and Feng Qi<sup>2,3</sup>

**Abstract:** In the paper the authors show that neither the function

$$\ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x + x - \frac{1}{2} \ln(2\pi) - a_i [(-1)^{i+1} \psi^{(i)}(x + b_i)]$$

nor its negative is completely monotonic on  $(0, \infty)$ , where  $a_i > 0$  and  $b_i \geq 0$  are real numbers,  $i \geq 2$ ,  $\Gamma(x)$  is the classical Euler's gamma function, and  $\psi^{(i)}(x) = \left[\frac{\Gamma'(x)}{\Gamma(x)}\right]^{(i)}$  are polygamma functions. Moreover, some other results are given in the form of remarks.

**Subjects:** Advanced Mathematics, Analysis - Mathematics, Integral Transforms & Equations, Mathematical Analysis, Mathematics & Statistics, Real Functions, Science, Special Functions

**Keywords:** complete monotonicity, gamma function, polygamma function

**AMS Subject Classifications:** Primary 26A48, 33B15, Secondary 44A10

### 1. Introduction

A function  $f(x)$  is said to be completely monotonic on an interval  $I$  if it has derivatives of all orders on  $I$  and satisfies

$$(-1)^n f^{(n)}(x) \geq 0 \tag{1.1}$$

### ABOUT THE AUTHORS

Bai-Ni Guo is a full Professor in Mathematics at Henan Polytechnic University and received the Bachelor degree in Mathematics for Education at Henan Normal University in China. Her research interests are in the areas of mathematical inequalities, mathematical means, special functions, combinatorics, number theory, and the like. She has published over 150 research articles in reputed international journals of mathematics. She is referees and editors of several mathematical journals.

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### PUBLIC INTEREST STATEMENT

In the paper, the authors deny the complete monotonicity of a class of functions involving the logarithm for the classical Euler gamma function and polygamma functions. A nonnegative and infinitely differentiable function is called a completely monotonic function if itself and all of its derivatives change their signs alternatively. A function is completely monotonic on the positive semi-axis if and only if it is a Laplace transform.



for  $x \in I$  and  $n \geq 0$ . For more information on this class of functions, please refer to Mitrinović, Pečarić, and Fink (1993, Chapter XIII) and Widder (1946, Chapter IV).

It is common knowledge that the classical Euler's gamma function is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt \tag{1.2}$$

for  $x > 0$ , that the logarithmic derivative of  $\Gamma(x)$  is called psi or digamma function and denoted by

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{1.3}$$

for  $x > 0$ , and that the derivatives  $\psi^{(i)}(x)$  for  $i \in \mathbb{N}$  and  $x > 0$  are called polygamma functions.

In Alzer and Batir (2007, p. 779), Alzer and Batir obtained that

(1) the function

$$G_a(x) = \ln \Gamma(x) - x \ln x + x - \frac{1}{2} \ln(2\pi) + \frac{1}{2} \psi(x+a) \tag{1.4}$$

for  $a \geq 0$  is completely monotonic on  $(0, \infty)$  if and only if  $a \geq \frac{1}{3}$ ,

(2) so is the function  $-G_a(x)$  if and only if  $a = 0$ .

We remark that the function  $G_a(x)$  may be reformulated as

$$G_a(x) = \ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x + x - \ln \sqrt{2\pi} + \frac{1}{2} [\psi(x+a) - \ln x] \tag{1.5}$$

In Merkle (1998, Theorem 1), Merkle proved that the function  $F_0(x)$  is strictly concave and the function  $F_a(x)$  for  $a \geq \frac{1}{2}$  is strictly convex on  $(0, \infty)$ , where

$$F_a(x) = \ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x - \frac{1}{12} \psi'(x+a) \tag{1.6}$$

for  $a \geq 0$ . See also Qi (2010, p. 46, Section 4.3.3).

Stimulated by Merkle (1998, Theorem 1), the authors considered in Qi (2015), its preprint Qi (2013), and Şevli and Batir (2011) the function

$$f_a(x) = \frac{1}{2} \ln(2\pi) - x + \left(x - \frac{1}{2}\right) \ln x - \ln \Gamma(x) + \frac{1}{12} \psi'(x+a) \tag{1.7}$$

for  $a \geq 0$  and established that

(1) the function  $f_a(x)$  is completely monotonic on  $(0, \infty)$  if and only if  $a = 0$ ,

(2) so is the function  $-f_a(x)$  if and only if  $a \geq \frac{1}{2}$ .

After analysing the functions (1.5) and (1.7), we naturally introduce the function

$$f_{a,b_i}(x) = \ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x + x - \frac{1}{2} \ln(2\pi) - a_i [(-1)^{i+1} \psi^{(i)}(x+b_i)] \tag{1.8}$$

on  $(0, \infty)$ , where  $a_i > 0$  and  $b_i \geq 0$  are real numbers and  $i \geq 2$  are integers, and instinctively ask for complete monotonicity of it.

The main result of this paper is that neither  $f_{a_i, b_i}(x)$  nor  $-f_{a_i, b_i}(x)$  is completely monotonic on  $(0, \infty)$ , which can be formulated as the following theorem.

**THEOREM 1.1** For all real numbers  $a_i > 0$  and  $b_i \geq 0$  and all integers  $i \geq 2$ , neither the function  $f_{a_i, b_i}(x)$  defined by (1.8) nor its negative is completely monotonic on  $(0, \infty)$ .

In the final section of this paper, we also give several remarks about complete monotonicity of  $f_{a_i, b_i}(x)$  and other functions related to the gamma function.

## 2. Proof of Theorem 1.1

The idea of this proof comes from the second proof of Qi (2013, Theorem 3.1) and its formally published version in Qi (2015, Theorem 2).

The famous Binet's first formula of  $\ln \Gamma(x)$  for  $x > 0$  is given by

$$\ln \Gamma(x) = \left(x - \frac{1}{2}\right) \ln x - x + \frac{1}{2} \ln(2\pi) + \theta(x) \tag{2.1}$$

where

$$\theta(x) = \int_0^\infty \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2}\right) \frac{e^{-xt}}{t} dt \tag{2.2}$$

for  $x > 0$  is called the remainder of Binet's first formula for the logarithm of the gamma function  $\Gamma(x)$ . See Magnus (1966, p. 11) or Qi and Guo (2010, p. 462). Combining this with the integral representation

$$\psi^{(k)}(x) = (-1)^{k+1} \int_0^\infty \frac{t^k}{1 - e^{-t}} e^{-xt} dt \tag{2.3}$$

for  $x > 0$  and  $k \in \mathbb{N} = \{1, 2, \dots\}$ , see Abramowitz and Stegun (1972, p. 260, 6.4.1), yields

$$\begin{aligned} f_{a_i, b_i}(x) &= \theta(x) - a_i [(-1)^{i+1} \psi^{(i)}(x + b_i)] \\ &= \int_0^\infty \left[ \frac{1}{t} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) - \frac{a_i t^i e^{-b_i t}}{1 - e^{-t}} \right] e^{-xt} dt \\ &= \int_0^\infty \left[ \frac{1 - e^{-t}}{t^{i+1}} \left( \frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) - a_i e^{-b_i t} \right] \frac{t^i e^{-xt}}{1 - e^{-t}} dt \\ &= \int_0^\infty \left[ \frac{1}{t^{i+1}} \left( \frac{e^{-t} + 1}{2} + \frac{e^{-t} - 1}{t} \right) - a_i e^{-b_i t} \right] \frac{t^i e^{-xt}}{1 - e^{-t}} dt \end{aligned}$$

For  $t > 0$  and  $i \geq 2$ , let

$$h_i(t) = \frac{1}{t^{i+1}} \left( \frac{e^{-t} + 1}{2} + \frac{e^{-t} - 1}{t} \right) \tag{2.4}$$

Completely monotonic functions were characterized by Widder (1946, p. 160, Theorem 12a) which reads that a necessary and sufficient condition that  $f(x)$  should be completely monotonic in  $0 \leq x < \infty$  is that

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t) \tag{2.5}$$

where  $\alpha(t)$  is bounded and non-decreasing and the integral converges for  $0 \leq x < \infty$ . Therefore, in order to prove complete monotonicity of  $f_{a_i, b_i}(x)$  on  $(0, \infty)$ , it suffices to show the positivity or negativity of the function  $h_i(t) - a_i e^{-b_i t}$  on  $(0, \infty)$ , which is equivalent to

$$h_i(t) \geq a_i e^{-b_i t}, \quad \frac{1}{t} \ln \frac{h_i(t)}{a_i} \geq -b_i, \quad \frac{1}{t} \ln \frac{h_i(t)}{a_i} \geq -b_i, b_i \geq -\frac{1}{t} \ln \frac{h_i(t)}{a_i}$$

Because

$$\begin{aligned} \frac{1}{t} \ln \frac{h_i(t)}{a_i} &= \frac{1}{t} \ln \frac{h_1(t)}{a_i t^{i-1}} = \frac{\ln[12h_1(t)]}{t} - \frac{\ln(12a_i t^{i-1})}{t} \\ \frac{\ln[12h_1(t)]}{t} &\rightarrow \begin{cases} -\frac{1}{2}, & t \rightarrow 0^+, \\ 0, & t \rightarrow \infty, \end{cases} \quad \text{and} \quad \frac{\ln(12a_i t^{i-1})}{t} \rightarrow \begin{cases} -\infty, & t \rightarrow 0^+ \\ 0, & t \rightarrow \infty \end{cases} \end{aligned} \quad (2.6)$$

it follows that

$$\frac{1}{t} \ln \frac{h_i(t)}{a_i} \rightarrow \begin{cases} -\infty, & t \rightarrow 0^+ \\ 0, & t \rightarrow \infty \end{cases}$$

for  $i \geq 2$ . Thus, if  $b_i > 0$  the function  $h_i(t) - a_i e^{-b_i t}$  for  $i \geq 2$  and any positive number  $a_i$  is neither constantly positive nor constantly negative on  $(0, \infty)$ , so, by Widder (1946, p. 160, Theorem 12a) mentioned above, neither the function  $f_{a_i, b_i}(x)$  for  $i \geq 2$ ,  $a_i > 0$  and  $b_i > 0$  nor its negative is completely monotonic on  $(0, \infty)$ .

It is easy to calculate that

$$\lim_{t \rightarrow 0^+} h_i(t) = \infty \quad \text{and} \quad \lim_{t \rightarrow \infty} h_i(t) = 0$$

for  $i \geq 2$ . Hence, if  $b_i = 0$  and  $a_i > 0$ , the function  $h_i(t) - a_i$  for  $i \geq 2$  is still neither constantly positive nor constantly negative on  $(0, \infty)$ . Consequently, neither the function  $f_{a_i, 0}(x)$  for  $i \geq 2$  and  $a_i > 0$  nor its negative is completely monotonic on  $(0, \infty)$ . The proof of Theorem 1.1 is complete.

### 3. Remarks

In this section, we list several remarks about some functions related to the logarithm of the gamma function  $\Gamma(x)$ .

*Remark 3.1* For  $a_i \leq 0$  and  $b_i \geq 0$ , how about complete monotonicity of the function  $f_{a_i, b_i}(x)$  on  $(0, \infty)$ ?

Since the function

$$\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \quad (3.1)$$

is positive and increasing on  $(0, \infty)$ , see Guo and Qi (2009; 2010), Zhang, Guo, and Qi (2009) and closely related references therein, the remainder  $\theta(x)$ , defined by (2.2), of Binet's first formula for the logarithm of the gamma function  $\Gamma(x)$  is clearly completely monotonic on  $(0, \infty)$ . The formula (2.3) reveals that the functions  $(-1)^{i+1} \psi^{(i)}(x)$  are completely monotonic on  $(0, \infty)$ . As a result of the facts that the sum of finitely many completely monotonic functions and the product of any positive number of completely monotonic functions are both completely monotonic, we obtain that for all real numbers  $a_i \leq 0$  and  $b_i \geq 0$  the function  $f_{a_i, b_i}(x)$  defined by (1.8) is trivially completely monotonic on  $(0, \infty)$ .

*Remark 3.2* Is the constant  $\frac{1}{12}$  in (1.7) the best possible? I believe it is.

Repeating the process in the proof of Theorem 1.1 for  $i = 1$ , we obtain that

(1) if  $a_1 > \frac{1}{12}$ ,

$$\frac{1}{t} \ln \frac{h_1(t)}{a_1} = \frac{\ln[12h_1(t)]}{t} - \frac{\ln(12a_1)}{t} \rightarrow \begin{cases} -\infty, & t \rightarrow 0^+ \\ 0, & t \rightarrow \infty \end{cases} \quad (3.2)$$

(2) if  $a_1 < \frac{1}{12}$ ,

$$\frac{1}{t} \ln \frac{h_1(t)}{a_1} = \frac{\ln[12h_1(t)]}{t} - \frac{\ln(12a_1)}{t} \rightarrow \begin{cases} \infty, & t \rightarrow 0^+ \\ 0, & t \rightarrow \infty \end{cases} \quad (3.3)$$

Combining these with (2.6), we see that only the function  $f_{a_1, b_1}(x)$  for  $a_1 \leq \frac{1}{12}$  and  $b_1 \geq 0$  is possible to be completely monotonic on  $(0, \infty)$ . It is apparent that

$$f_{a_1, b_1}(x) = f_{1/12, b_1}(x) + \left(\frac{1}{12} - a_1\right) \psi'(x + b_1)$$

This means that, for  $a_1 \leq \frac{1}{12}$  and  $b_1 \geq \frac{1}{2}$ , the function  $f_{a_1, b_1}(x)$  is completely monotonic on  $(0, \infty)$ . Consequently, since  $f_{1/12, b_1} = -f_{b_1}(x)$ , the coefficient  $\frac{1}{12}$  in (1.7) is the best possible in the sense that the scalar  $\frac{1}{12}$  cannot be replaced by a bigger one. This sharpens the main result in Qi (2013; 2015), Şevli and Batir (2011).

*Remark 3.3* Is the scalar  $\frac{1}{2}$  in the last term of (1.5) the best possible? I believe it is.

For  $a \geq 0$  and  $b \neq 0$ , let

$$G_{a,b}(x) = \theta(x) + b[\psi(x+a) - \ln x] \quad (3.4)$$

on  $(0, \infty)$ . It is clear that  $G_{a, 1/2}(x) = G_a(x)$ . An easy calculation leads to

$$\begin{aligned} G'_{a,b}(x) &= \theta'(x) + b \left[ \psi'(x+a) - \frac{1}{x} \right] \\ &= \int_0^\infty \left[ b \left( \frac{te^{-at}}{1-e^{-t}} - 1 \right) - \left( \frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} \right) \right] e^{-xt} dt \\ &= \int_0^\infty \left\{ 2b \frac{t[te^{(1-a)t} - e^t + 1]}{(t-2)e^t + t + 2} - 1 \right\} \left( \frac{1}{e^t-1} - \frac{1}{t} + \frac{1}{2} \right) e^{-xt} dt \end{aligned}$$

Let

$$q_a(t) = \frac{t[te^{(1-a)t} - e^t + 1]}{(t-2)e^t + t + 2}$$

for  $a \geq 0$  and  $t > 0$ . By L'Hôspital rule, we have

$$\begin{aligned} \lim_{t \rightarrow \infty} q_a(t) &= - \lim_{t \rightarrow \infty} \frac{e^{-at} \{ (t+1)e^{(1+a)t} - e^{at} + te^t [(a-1)t - 2] \}}{(t-1)e^t + 1} \\ &= - \lim_{t \rightarrow \infty} \frac{e^{-at} [(t+2)e^{at} - (a-1)^2 t^2 + 4(a-1)t - 2]}{t} \\ &= - \lim_{t \rightarrow \infty} \{ 1 + [a^3 t^2 - 2a^2 t(t+3) + a(t^2 + 8t + 6) - 2(t+2)] e^{-at} \} \\ &= \begin{cases} -1, & a > 0 \\ \infty, & a = 0 \end{cases} \end{aligned}$$

and  $\lim_{t \rightarrow 0^+} q_a(t) = 3(1-2a)$ . For the function  $G_{a,b}(x)$  to be completely monotonic on  $(0, \infty)$ , it is necessary that  $6b(1-2a) - 1 \leq 0$  which implies that  $a < \frac{1}{2}$  and  $b \leq \frac{1}{6(1-2a)}$  are both necessary. Since  $\frac{1}{6(1-2a)}$  has a minimum  $\frac{1}{2}$  for  $\frac{1}{3} \leq a < \frac{1}{2}$  and  $G_{1/2, b}(x)$  is completely monotonic on  $(0, \infty)$  if and only if  $a \geq \frac{1}{3}$ , the coefficient  $\frac{1}{2}$  in front of the first bracket in (1.5) is the best possible in the sense that it cannot be substituted by a larger number.

**Remark 3.4** Now we outline an alternative proof of the main result in Alzer and Batir (2007, p. 779).

When  $b = \frac{1}{2}$ , the function  $G_{a,1/2}(x) = G_a(x)$ . In order to prove that  $G_{a,1/2}(x)$  or its negative is completely monotonic on  $(0, \infty)$ , it is necessary that the function  $-G_{a,1/2}(x)$  or its negative is completely monotonic on  $(0, \infty)$ . The latter is equivalent to the inequality  $q_a(t) \leq 1$  on  $(0, \infty)$  and this inequality may be rewritten as

$$1 - a \leq \frac{1}{t} \ln \frac{2[(t-1)e^t + 1]}{t^2}$$

The function

$$\frac{1}{t} \ln \frac{2[(t-1)e^t + 1]}{t^2}$$

is strictly increasing on  $(0, \infty)$ , with limits

$$\lim_{t \rightarrow 0^+} \frac{1}{t} \ln \frac{2[(t-1)e^t + 1]}{t^2} = \frac{2}{3} \quad \text{and} \quad \lim_{t \rightarrow \infty} \frac{1}{t} \ln \frac{2[(t-1)e^t + 1]}{t^2} = 1$$

This implies that the function  $G_{a,1/2}(x)$  is completely monotonic on  $(0, \infty)$  if and only if  $a \geq \frac{1}{3}$  and so is the function  $-G_{a,1/2}(x)$  if and only if  $a = 0$ .

**Remark 3.5** By the way, some complete monotonicity properties of functions involving the gamma and polygamma functions and relating to differences between two remainders  $\theta(x)$  of Binet's first formula for the logarithm of the gamma function  $\Gamma(x)$  have been investigated in Chen, Qi, and Srivastava (2010, p. 162–163, Section 5), Qi and Guo (2010, p. 464, Section 1.4, Theorem 3) and Chen and Srivastava (2011), Chen and Srivastava, Li, and Manyama (2011), Guo, Qi, and Srivastava (2010), Guo and Qi (2009), Guo, Qi, and Srivastava (2012), Guo, Qi, and Srivastava (2007; 2008), Guo and Srivastava (2009; 2008), Qi, Niu, and Guo (2007), Srivastava, Guo, and Qi (2012). For more information, please refer to the survey articles Qi (2010), Qi and Luo (2012) and plenty of references therein.

**Remark 3.6** Finally, we pose an open problem. Discuss complete monotonicity of the functions

$$\ln \Gamma(x) - \left(x - \frac{1}{2}\right) \ln x + x - \frac{1}{2} \ln(2\pi) - a_i \left[ (-1)^i \psi^{(i)}(x + b_i) + \frac{(i-1)!}{x^i} \right] \tag{3.5}$$

and their negatives on  $(0, \infty)$ , where  $a_i > 0$  and  $b_i \geq 0$  are real numbers and  $i \geq 1$  are integers.

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**References**

Abramowitz, M., & Stegun, I. A. (Eds.). (1972). *Mathematical functions with formulas, graphs, and mathematical tables*. Vol. 55, applied mathematics series, Tenth printing, with corrections. Washington, DC: National Bureau of Standards.

Alzer, H., & Batir, N. (2007). Monotonicity properties of the gamma function. *Applied Mathematics Letters*, 20, 778–781. Retrieved from <http://dx.doi.org/10.1016/j.aml.2006.08.026>

Chen, C.-P., Qi, F., & Srivastava, H. M. (2010). Some properties of functions related to the gamma and psi functions. *Integral Transforms and Special Functions*, 21, 153–164. Retrieved from <http://dx.doi.org/10.1080/10652460903064216>

Chen, C.-P., & Srivastava, H. M. (2011). Some inequalities and monotonicity properties associated with the gamma and psi functions and the Barnes G-function. *Integral Transforms and Special Functions*, 22, 1–15. Retrieved from <http://dx.doi.org/10.1080/10652469.2010.483899>

Chen, C.-P., Srivastava, H. M., Li, L., & Manyama, S. (2011). Inequalities and monotonicity properties for the psi



- (or digamma) function and estimates for the Euler-Mascheroni constant. *Integral Transforms and Special Functions*, 22, 681–693. Retrieved from <http://dx.doi.org/10.1080/10652469.2010.538525>
- Guo, B.-N., Qi, F., & Srivastava, H. M. (2010). Some uniqueness results for the non-trivially complete monotonicity of a class of functions involving the polygamma and related functions. *Integral Transforms and Special Functions*, 21, 849–858. Retrieved from <http://dx.doi.org/10.1080/10652461003748112>
- Guo, S., & Qi, F. (2009). A class of completely monotonic functions related to the remainder of Binet's formula with applications. *Tamsui Oxford Journal of Mathematical Sciences*, 25, 9–14.
- Guo, S., Qi, F., & Srivastava, H. M. (2012). A class of logarithmically completely monotonic functions related to the gamma function with applications. *Integral Transforms and Special Functions*, 23, 557–566. Retrieved from <http://dx.doi.org/10.1080/10652469.2011.611331>
- Guo, S., Qi, F., & Srivastava, H. M. (2007). Necessary and sufficient conditions for two classes of functions to be logarithmically completely monotonic. *Integral Transforms and Special Functions*, 18, 819–826. Retrieved from <http://dx.doi.org/10.1080/10652460701528933>
- Guo, S., Qi, F., & Srivastava, H. M. (2008). Supplements to a class of logarithmically completely monotonic functions associated with the gamma function. *Applied Mathematics and Computation*, 197, 768–774. Retrieved from <http://dx.doi.org/10.1016/j.amc.2007.08.011>
- Guo, S., & Srivastava, H. M. (2009). A certain function class related to the class of logarithmically completely monotonic functions. *Mathematical and Computer Modelling*, 49, 2073–2079. Retrieved from <http://dx.doi.org/10.1016/j.mcm.2009.01.002>
- Guo, S., & Srivastava, H. M. (2008). A class of logarithmically completely monotonic functions. *Applied Mathematics Letters*, 21, 1134–1141. Retrieved from <http://dx.doi.org/10.1016/j.aml.2007.10.028>
- Magnus, W., Oberhettinger, F., & Soni, R. P. (1966). *Formulas and theorems for the special functions of mathematical physics*. Berlin: Springer. Retrieved from <http://dx.doi.org/10.1216/rmj/1181071755>
- Merkle, M. (1998). Convexity, Schur-convexity and bounds for the gamma function involving the digamma function. *Rocky Mountain Journal of Mathematics*, 28, 1053–1066.
- Mitrinović, D. S., Pečarić, J. E., & Fink, A. M. (1993). *Classical and new inequalities in analysis*. Dordrecht: Kluwer Academic Publishers.
- Qi, F. (2010). Bounds for the ratio of two gamma functions. *Journal of Inequalities and Applications*. Article ID 493058. 84 pp. Retrieved from <http://dx.doi.org/10.1155/2010/493058>
- Qi, F. (2013). A completely monotonic function involving the gamma and tri-gamma functions. Retrieved from <http://arxiv.org/abs/1307.5407>
- Qi, F. (2015, in press). A completely monotonic function involving the gamma and tri-gamma functions. *Applied Mathematics & Information Sciences*, 8.
- Qi, F., & Guo, B.-N. (2010). Some properties of extended remainder of Binet's first formula for logarithm of gamma function. *Mathematica Slovaca*, 60, 461–470. Retrieved from <http://dx.doi.org/10.2478/s12175-010-0025-7>
- Qi, F., & Luo, Q.-M. (2012). Bounds for the ratio of two gamma functions—From Wendel's and related inequalities to logarithmically completely monotonic functions. *Banach Journal of Mathematical Analysis*, 6, 132–158.
- Qi, F., Niu, D.-W., & Guo, B.-N. (2007). Monotonic properties of differences for remainders of psi function. *International Journal of Pure and Applied Mathematical Sciences*, 4, 59–66.
- Şevli, H., & Batir, N. (2011). Complete monotonicity results for some functions involving the gamma and polygamma functions. *Mathematical and Computer Modelling*, 53, 1771–1775. Retrieved from <http://dx.doi.org/10.1016/j.mcm.2010.12.055>
- Srivastava, H. M., Guo, S., & Qi, F. (2012). Some properties of a class of functions related to completely monotonic functions. *Computers & Mathematics with Applications*, 64, 1649–1654. Retrieved from <http://dx.doi.org/10.1016/j.camwa.2012.01.016>
- Widder, D. V. (1946). *The Laplace transform*. Princeton, NJ: Princeton University Press.
- Zhang, S.-Q., Guo, B.-N., & Qi, F. (2009). A concise proof for properties of three functions involving the exponential function. *Applied Mathematics E-Notes*, 9, 177–183.



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